Floer Homotopy Theory

Notes for the *Séminaire de Mathématiques Supérieures 2022*

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These are notes for four lectures given by Mohammed Abouzaid at the PIMS Floer Homotopy Summer School and solutions to (some of) the exercises. Material from §2 onwards treats joint work of Mohammed Abouzaid and Andrew Blumberg. All mistakes are due to the author.[1]

Contents

[1] I thank Colin Fourel and Noah Porcelli for pointing some out.

1 Classical theory

Sucesses of Morse and Floer homology

Bott: Compute the homology of $O(n)$ in the limit $n \to \infty$ using

a) a choice of Riemannian metric g_n and

b) a choice of norm function $f_n\colon O(n) \to \mathbb{R}$ related to the height function on S^n.

The critical points of this function form submanifolds. One then studies the solutions of the flow equation determined by g_n and f_n, i.e., $\gamma\colon \mathbb{R} \to O(n)$ such that $\dot{\gamma} = -\nabla f_n(\gamma)$. Denote

$$\mathcal{M}(x, y) = \{\gamma : \lim_{t\to\infty} \gamma(t) = x,\ \lim_{t\to-\infty} \gamma(t) = y\}/\mathbb{R}$$

From an axiomatic perspective on homology, this looks strange.

Floer: If L is a closed smooth manifold and $L' \subset T^*L$ is Lagrangian isotopic to L, then (after a generic perturbation) we have the lower bound $|L \cap L'| \geq \operatorname{rank}(H_*(L))$.
This is done by choosing an almost complex structure J on T^*L and studying the gradient flows of the action functional on

$$\mathcal{P}(L, L') := \{\gamma \in C^\infty([0, 1], T^*L) : \gamma(0) \in L,\ \gamma(1) \in L'\}.$$

The critical points in this case are intersections of L and L' and the differential is given by counting J-holomorphic strips with boundary on L and L' that converge to intersection points at the respective ends.

1.1 Bordism theory

There are two basic choices to make: which class of manifolds (Top, PL, Diff) and which structure on them (O, SO, U, Spin, $\dots$)? Unless otherwise mentioned, we work with smooth oriented manifolds in these notes, i.e., with $\Omega^{\mathrm{Diff},SO}$, although we will denote it by Ω for simplicity.

For details, see [CF62].

Definition 1.1. Let (X, A) be a topological pair. An *singular manifold* (M, f) in (X, A) consists of an oriented smooth compact manifold M with boundary and a continuous map $f\colon M \to X$ such that $f(\partial M) \subset X$.

Definition 1.2. Two singular manifolds (M, f) and (M', f') of dimension k are *bordant* if there exists an oriented manifold W^{k+1} with corners and a continuous map $F\colon W \to X$ together with embeddings $M \hookrightarrow \partial W$ (orientation-preserving) and $M' \hookrightarrow \partial W$ (oriented-reversing) such that

$$\partial W \subset M \sqcup M' \cup F^{-1}(A).$$

Remark 1.3. Being bordant is an equivalence relation. Reflexivity and symmetry are immediate while transitivity follows from a connected sum argument (and smoothing the corners), e.g. [MS17, Exercise 7.1.4] and [CF62, Chapter I.3].

Definition 1.4. The *kth (relative) bordism group* of (X, A) is

$$\Omega_k(X, A) := \left\{(M, f) : \substack{M^k \text{ smooth oriented compact manifold with boundary} \\ f\colon (M,\partial M)\to(X,A) \text{ continuous}}\right\} / \sim .$$

Here $(M, f) \sim (M', f')$ if they are *bordant*. We denote by Ω_k the bordism group of the point in dimension k.

Remark 1.5. Alternatively, one can define a bordism between M_0 and M_1 to be a manifold W with corners endowed with a smooth map $\pi\colon W \to [0, 1]$ and embeddings $M_i \hookrightarrow \pi^{-1}(\{i\})$ such that $\pi \pitchfork \{0, 1\}$ with $\pi^{-1}(\{0, 1\}) = \partial W$ (with the appropriate orientations).

The set $\Omega_k(X, A)$ has a natural additive structure induced by the disjoint sum with neutral element the class of $\emptyset$ and the inverse given by reversing the orientation of the manifold. We denote $\Omega_*(X, A) := \bigoplus_{k\geq 0} \Omega_k(X, A)$.

Lemma 1.6. *Let $j\colon A \hookrightarrow X$ and $i\colon (X, \emptyset) \to (X, A)$ be the inclusions. There exists a natural long exact sequence*

$$\ldots \to \Omega_k(A) \xrightarrow{j_*} \Omega_k(X) \xrightarrow{i_*} \Omega_k(X, A) \xrightarrow{\partial} \Omega_{k-1}(A) \to \ldots$$

Proof. The map ∂ is defined by setting $\partial[M, f] = [\partial M, f|_{\partial M}]$, the restriction considered as a map to A. Clearly, the composition of any two maps in the sequence is zero. Suppose $\partial[M, f] = 0$. Then there exists a smooth manifold W^k with boundary, an (orientation-preserving) embedding $\iota\colon \partial M \hookrightarrow \partial W$ and $F\colon W \to A$ such that $\partial W = \iota(M)$. Let $\tilde{N} := W\#_{\partial M} M$ be the oriented connected sum, and let $\tilde{g}\colon \tilde{N} \to X$ be induced by F on W and f on M, using a reparametrisation on a collar of $\partial M = \partial W$ to patch the two functions together continuously. Then $[\tilde{N}, \tilde{g}] \in \Omega_k(X)$. To see that $[M, f] = i_*[\tilde{N}, \tilde{g}]$, set $N := \tilde{N} \times [0, 1]$ (with the canonical orientation) and let $g := \tilde{g}\,\mathrm{pr}_1$. There are canonical embeddings $M \hookrightarrow \tilde{N} \times \{0\} \subset \partial N$ and $\tilde{N} \hookrightarrow \tilde{N} \times \{1\} \subset \partial N$, the first orientation-preserving, the second orientation-reversing. As $W \times \{0\} \subset g^{-1}(A)$, exactness at $\Omega_k(X, A)$ follows.
If $j_*[M, f] = 0$ for $[M, f] \in \Omega_k(A)$, then there exists $F\colon W \to X$ with W^{k+1} such that M embeds surjectively onto ∂W and $F|_M = f$. Thus, $[M, f] \in i_*\Omega_{k+1}(X, A)$. Finally, if $i_*[M, f] = 0$, let $F\colon W^{k+1} \to X$ be a bordism from (M, f) to $\emptyset$ in (X, A). Then $N := \partial W \setminus M \subset F^{-1}(A)$ is a union of path components of ∂W as M is open and closed in ∂W. Therefore, (W, F) is a bordism from (M, f) to $(N, F|_N)$ (endowed with the opposite orientation). Exactness at $\Omega_k(X)$ follows as well. □

Lemma 1.7. *If $U \subset X$ is a set such that $\overline{U} \subset A^{\mathrm{o}}$, then $\Omega_*(X \setminus U, A \setminus U) \to \Omega_*(X, A)$ is an isomorphism.*

Lemma 1.8. *Given two topological pairs (X, A) and (Y, B) there exists a natural multiplication map*

$$\Omega_k^{\mathrm{Top}}(X, A) \times \Omega_\ell^{\mathrm{Top}}(Y, B) \to \Omega_{k+\ell}^{\mathrm{Top}}(X \times Y, X \times B \cup A \times Y).$$

Proof. We first show that a manifold with corners is homeomorphic to a manifold with boundary but without corners. Using permutation of coordinates and induction, it suffices to show that $(\mathbb{R}_{\geq 0})^2$ is homeomorphic to $\mathbb{R} \times \mathbb{R}_{\geq 0}$. For this, identify $\mathbb{R}^2$ with $\mathbb{C}$ and apply the map $z \mapsto z^2$. If $f\colon (M, \partial M) \to (X, A)$ and $g\colon (N, \partial N) \to (Y, B)$ are singular manifolds, then

$$f \times g\colon (M \times N, \partial(M \times N)) \to (X \times Y, X \times B \cup A \times Y)$$

is singular manifold by the first observation. Thus, we can set

$$[(f, M)] \cdot [(g, N)] := [(f \times g, M \times N)].$$

This is a well-defined operation on the equivalence classes by a connected sum argument as in the next proof. □

Corollary 1.9. *Ω_* is a graded commutative ring, and $\Omega_*(X, A)$ is an Ω_*-module for any topological pair (X, A).*

Proof. Suppose M and N are oriented smooth manifolds and W is an oriented bordism from M to M' and Q an oriented bordism from N to N'. Then $W \times N$ is an oriented bordism from $M \times N$ to $M' \times N$ and $M' \times Q$ is one from $M' \times N$ to $M' \times N'$. By the transitivity of the bordism relation, it follows that $M \times N$ is (oriented) bordant to $M' \times N'$. For the last assertion, we set $[N] \cdot [M, f] := [N \times M, f\, \mathrm{pr}_2]$ for $[N] \in \Omega_*$ and $[M, f] \in \Omega_*(X, A)$. Well-definedness is a straightforward verification. □

1.2 Gluing of manifolds

Let us record a few statements about the gluing of smooth manifolds, which will be useful later on.

Definition 1.10. Let M be a smooth manifold with corners. A *collar* of the codimension k boundary $\partial^k M$ is a smooth embedding $h\colon \partial M \times [0, 1]^k \to M$ such that $h(\cdot, 0)$ is the inclusion of ∂M. We denote the space of smooth collars of the codimension 1 boundary by $\mathrm{Collar}(M)$.

Lemma 1.11 (Exercise 3.2)**.** *Let M be a smooth manifold with boundary. Then* $\mathrm{Collar}(M)$ *with the subpsace topology from $C^\infty(\partial M \times [0, 1], M)$ is a contractible space.*

Proof. Suppose first $M = \partial M \times \mathbb{R}_{\geq 0}$ and let

$$\mathrm{Collar}_{\mathrm{lin}}(M) := \{f \in \mathrm{Collar}(M) : \forall x \in \partial M \forall t \in [0, 1] : f(x, t) = (x, a(x)t)\}.$$

Then we can define $H\colon \operatorname{Collar}(M)\times[0,1]\to\operatorname{Collar}(M)$ by

$$H(f,s)(x,t)=\begin{cases}(f_1(x,st),\frac{1}{s}f_2(x,st)) & s>0\\ (x,\partial_2 f_2(x,0)t) & s=0\end{cases}$$

to obtain a deformation retraction onto $\operatorname{Collar}_{\mathrm{lin}}(M)$. A contraction of $\operatorname{Collar}_{\mathrm{lin}}(M)$ is given by

$$F(f,s)(x,t)=(x,(1-s)f_2(x,t)+st).$$

In the general case, fix a Riemannian metric g on M with distance d and let $\psi\colon \partial M\times[0,2)\to U\subset M$ be a collar of ∂M such that $d(x,\partial U)=2$ for any $x\in\partial M$. Using the paracompactness of ∂M and $C^\infty(\partial M\times[0,1],M)$, we obtain a continuous function $\delta\colon \operatorname{Collar}(M)\times\partial M\to(0,1]$ which is smooth in the second factor such that $f(\{x\}\times[0,\delta(f,x)])\subset U$ for $x\in\partial M$ and f a collar of ∂M in M. Then define $G\colon \operatorname{Collar}(M)\times[0,1]\to\operatorname{Collar}(M)$ by

$$G(f,s)(x,t)=f(x,(1-s)t+s\delta(f,x)t).$$

Then $G_1(f)$ has image in U and so we can apply the argument of the first step to the image of $f\mapsto\psi^{-1}G_1(f)$ to obtain the claim. $\square$

Lemma 1.12 (Exercise 3.3)**.** *Let M be a smooth manifold with boundary. Then there exists a smooth map $H\colon \operatorname{Collar}(M)\times[0,1]\to\operatorname{Diff}(M)$ such that $H_0(\psi)=\mathrm{id}$ for any $\psi\in\operatorname{Collar}(M)$ and $H_1(\psi)\circ\psi$ is the constant map at some collar of ∂M.*

Proposition 1.13 (Exercise 3.4)**.** *Suppose M_1 and M_2 are smooth manifolds with boundary and there exists a diffeomorphism $\phi\colon\partial M_1\to\partial M_2$. Then*

$$M_1\#_\partial M_2:=M_1\cup_\phi M_2$$

admits a smooth structure so that M_1 and M_2 embed smoothly and $T(M_1\#_\partial M_2)=TM_1\#_\partial TM_2$. This property characterises the smooth structure uniquely up to a contractible choice of isotopy.

Proof. Fix a collar $h_i\colon[0,1)\times\partial M_i\to U_i\subset M_i$ for $i\in\{1,2\}$. Define $\psi\colon(-1,1)\times M_1\to M_1\#_\partial M_2$ by

$$\psi(t,x)=\begin{cases}h_1(-t,x) & t\le 1\\ h_2(t,\phi(x)) & t\ge 0\end{cases}\tag{1.2.1}$$

and declare ψ to be a diffeomorphism. This defines a smooth structure on $U_1\#_\partial U_2$, which agrees with the smooth structure on U_i coming from M_i because h_1,h_2 and ϕ are diffeomorphisms. As $h_i^*TM_i=\mathbb{R}\times T\partial M_i$, the claim about tangent bundles is immediate. By Lemma 1.12, the choice of h_1 and h_2 is unique up to a contractible choice of isotopy. $\square$

Corollary 1.14. *i) If M_1 and M_2 are compact, $M_1\#_\partial M_2$ is closed.*

ii) If M_1 and M_2 are connected, so is $M_1\#_\partial M_2$.

iii) If M_1 and M_2 are oriented and ϕ is orientation-reversing, $M_1 \#_\partial M_2$ is oriented.

iv) If M_1 and M_2 are framed, then so is $M_1 \#_\partial M_2$.

v) Let $f_1 \colon M_1 \to N$ and $f_2 \colon M_2 \to N$ be smooth maps to another manifold with $f_2\phi = f_1|_{\partial M_1}$. If there exist inward-pointing normal vector fields $v_i \colon \partial M_i \to TM_i$ so that $\mathcal{L}_{v_1} f_1 = -\phi^*(\mathcal{L}_{v_2} f_2)$, then there exists a unique smooth map $f \colon M_1 \#_\partial M_2 \to N$ restricting to f_i on M_i.

Proof. i) Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $M_1 \#_\partial M_2$. Passing to a sequence, we may assume $(x_n)_{n\in\mathbb{N}} \subset M_i$ for some i. By compactness of M_i, $(x_n)_{n\in\mathbb{N}}$ has a convergent sequence. Since $M_i \hookrightarrow M_1 \#_\partial M_2$ is an embedding, this sequence converges in $M_1 \#_\partial M_2$.

ii) If M_1 and M_2 are connected, so is $M_1 \#_\partial M_2$.

iii) If M_1 and M_2 are oriented and ϕ is orientation-reversing, $M_1 \#_\partial M_2$ is oriented.

iv) Immediate.

v) Since $M_1 \#_\partial M_2$ is the pushout of $M_1 \hookleftarrow \partial M_1 \hookrightarrow M_2$, there exists a unique continuous map $f \colon M_1 \#_\partial M_2 \to N$ which restricts to f_i on M_i. Construct $M_1 \#_\partial M_2$ by using collars h_1, h_2 with $\partial_t h_i = v_i(h_i)$. Then $\partial_t f \psi$ is continuous, as is any higher derivative with respect to t. This completes the proof.

□

Remark 1.15 (Partial boundary connect sum). Suppose M_1, M_2 are smooth manifolds of dimension n with boundary and N is another smooth manifold of dimension $n-1$ admitting smooth embeddings $j_i \colon N \hookrightarrow \partial M_i$. Then we can construct $M_1 \#_N M_2$ as above to obtain a smooth manifold (possibly with boundary), which has properties analogous to those listed in Corollary 1.14.

Remark 1.16. We can consider the situation in Remark (1.15) as a special case of the more general setting. To this end, suppose M is a manifold with boundary and let $\phi \colon N_1 \to N_2$ be a diffeomorphism between two different boundary components. Then we can endow

$$M_\phi = M/\sim \qquad \text{where } x \sim y :\Leftrightarrow x = y \text{ or } x \in N_1, y = \phi(x)$$

with a smooth structure, which restricts to the smooth structure on $M \setminus (N_1 \cup N_2)$

Proposition 1.17 (Exercise 3.5)**.** *Suppose $k \geq 1$ and that for each nonempty $I \subset \{0, \dots, k\}$ we are given a smooth manifold M_I of dimension $n - |I| + 1$. Assume we have embeddings*

$$M_{I\cup\{j\}} \hookrightarrow M_I \tag{1.2.2}$$

of codimension 1 strata for $j \notin I$ such that all squares of the form

$$\begin{array}{ccc} M_{I\cup\{j,j'\}} & \longrightarrow & M_{I\cup\{j\}} \\ \downarrow & & \downarrow \\ M_{I\cup\{j'\}} & \longrightarrow & M_I \end{array}$$

commute. If all boundary strata of M_I are images of the embeddings (1.2.2)*, then the union of the manifolds M_i along the boundary strata has a smooth structure such that the embeddings of the M_I are smooth. This smooth structure is unique up to a contractible choice of ambient isotopies.*

2 Flow categories and flow modules

In the study of Morse or Floer homology we consider a functional with nondegenerate critical points. These freely generate a chain complex $\bigoplus_x \mathbb{Z}\langle x\rangle$ whose differential d is given by counting elements of $\mathcal{M}(x,y)$ where $\dim(\mathcal{M}(x,y)) = 0$. To obtain that $d^2 = 0$, the number $\langle d^2\langle x\rangle, \langle z\rangle\rangle$ is given by counting points in

$$\partial\mathcal{M}(x,z) \cong \bigsqcup_y \mathcal{M}(x,y) \times \mathcal{M}(y,z).$$

As Ω_k can be nonzero for $k > 0$, we have to consider higher-dimensional moduli spaces.

2.1 Flow categories

Notation 2.1. We denote by Man the category of (unoriented) smooth compact manifolds with corners with smooth maps between them. The full subcategory of oriented manifolds is denoted by $\mathrm{Man}_{\mathrm{or}}$.

The notion of a flow category was developed by Cohen, Jones, and Segal in [CJS95].

Definition 2.2. A *flow category* is a (non-unital) category $\mathcal{M}$ with finitely many objects enriched over $\mathrm{Man}_{\mathrm{or}}$ together with a degree function $|\cdot|\colon \mathrm{obj}(\mathcal{M}) \to \mathbb{Z}$ such that the following holds.

- $\dim(\mathcal{M}(x,y)) = |y| - |x| - 1$
- The composition $\mathcal{M}(x,y) \times \mathcal{M}(y,z) \to \mathcal{M}(x,z)$ is the smooth embedding of a codimension 1 boundary stratum of $\mathcal{M}(x,z)$ and

  $$\partial\mathcal{M}(x,z) = \bigcup_y \mathcal{M}(x,y) \times \mathcal{M}(y,z)$$

  where we identify the space with its image on the right hand side. The orientation of $\mathcal{M}(x,y) \times \mathcal{M}(y,z)$ agrees with the natural orientation of the boundary stratum up to a sign of $(-1)^{1+(|x|+1)(|z|-|y|)+|y||z|}$.

- Any codimension k stratum of $\mathcal{M}(x,z)$ is the image of an iterated composition map

$$\mathcal{M}(x,y_1)\times\mathcal{M}(y_1,y_2)\times\cdots\times\mathcal{M}(y_k,z)\hookrightarrow\mathcal{M}(x,z).$$

Remark 2.3. We want to define a notion of bordism theory $\Omega_*(\mathcal{M})$ on flow categories, which recovers $\Omega_*(X)$ if $\mathcal{M}$ is the flow category associated to a Morse function f on the closed oriented manifold X.

Lemma 2.4. *Suppose $\mathcal{M}$ is a flow category with a degree function $|\cdot|$. Then $\mathcal{M}^{opp}$ with the degree function $-|\cdot|$ is a flow category.*

Proof. Straightforward verification. □

Remark 2.5. We will usually omit the degree $|\cdot|$ from the notation. Whenever we talk about the opposite flow category, we endow it with the negative of the original degree function.

Definition 2.6. An *ungraded flow category* is a (non-unital) category $\mathcal{M}$ with finitely many objects enriched over Man such that the following holds.

- The composition $\mathcal{M}(x,y)\times\mathcal{M}(y,z)\to\mathcal{M}(x,z)$ is the smooth embedding of a codimension 1 boundary stratum of $\mathcal{M}(x,z)$ and

$$\partial\mathcal{M}(x,z)=\bigcup_y\mathcal{M}(x,y)\times\mathcal{M}(y,z).$$

- Any codimension k stratum of $\mathcal{M}(x,z)$ is the image of an iterated composition map.

Lemma 2.7. *Let $\mathcal{M}$ be an ungraded flow category. Then $\mathcal{M}(x,x)=\emptyset$ for any object x and there exists a partial order $\prec$ on $\mathrm{obj}(\mathcal{M})$ such that $x\prec y$ if and only if $\mathcal{M}(x,y)\neq\emptyset$.*

Proof. Suppose that $\mathcal{M}(x,x)$ is nonempty. The existence of $\mathcal{M}(x,x)\times\mathcal{M}(x,x)\hookrightarrow\mathcal{M}(x,x)$ implies that $\dim(\mathcal{M}(x,x))=0$. By compactness, $\mathcal{M}(x,x)$ is thus finite. The injectivity of the above composition map forces $|\mathcal{M}(x,x)|=0$. The verification of the second claim is straightforward. □

A concrete example from Morse theory

Let X be a smooth closed manifold and $f\colon X\to\mathbb{R}$ a Morse function. These data define a flow category $\mathcal{M}_f$ whose objects are the critical points of f and where the degree function is given by the Morse index of f. The space of morphisms $\mathcal{M}_f(p,q)$ is the compactification of the space of *positive* gradient flow lines from p to q

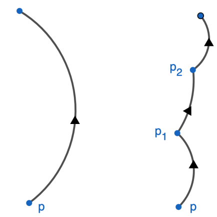
$$\{\gamma\in C^\infty(\mathbb{R},X):\dot\gamma=\nabla f(\gamma),\ \lim_{t\to-\infty}\gamma(t)=p,\ \lim_{t\to\infty}\gamma(t)=q\}$$

modulo $\mathbb{R}$-translations in the domain, given by "adding" broken Morse flow lines.

The *descending manifold* of $p \in \mathrm{Crit}(f)$ is $\mathcal{D}(p)$, the compactification of

$$\{\gamma \in C^\infty([0,\infty), X) : \dot\gamma = \nabla f(\gamma),\ \lim_{t\to\infty} \gamma(t) = p\}$$

given by adding broken flow lines "at the positive end". It is a manifold with corners of dimension $\mathrm{ind}_p(f)$. Its boundary strata are of the form



$$\mathcal{D}(p_k) \times \mathcal{M}_f(p_k, p_{k-1}) \times \cdots \times \mathcal{M}_f(p_1, p)$$

for critical points $p_1, \dots, p_k$ of f. The descending manifolds are an (important) example of a right flow module, a notion we introduce next.

2.2 Flow modules

Definition 2.8. A *right flow module* over a flow category $\mathcal{M}$ is a map $\mathcal{N}\colon \mathrm{obj}(\mathcal{M}) \to \mathrm{Man}_{\mathrm{or}}$ together with the data of smooth embeddings $\mathcal{N}(x)\times\mathcal{M}(x,y) \to \mathcal{N}(y)$ such that any codimension 1 stratum of $\partial\mathcal{N}(x)$ is of the form $\mathcal{N}(y) \times \mathcal{M}(y, x)$ with the orientations agreeing up to the sign $(-1)^{1+k(|y|-|x|)+|x||y|}$[2]. We abbreviate this by

$$\partial\mathcal{N}(y) = \bigcup_{|x|<|y|} \mathcal{N}(x) \times \mathcal{M}(x, y).$$

The boundary strata of codimension $\ell > 1$ are given by images of the embeddings

$$\mathcal{N}(x) \times \mathcal{M}(x, y_1) \times \cdots \times \mathcal{M}(y_{\ell-1}, y_\ell) \hookrightarrow \mathcal{N}(x).$$

We say that $\mathcal{N}$ has *dimension* k if $\dim(\mathcal{N}(y)) = k + |y|$ for $y \in \mathrm{obj}(\mathcal{M})$. Denote the set of right flow modules over $\mathcal{M}$ of dimension k by $\Theta_k\mathcal{M}$.

Slogan: $\mathcal{N}(x)$ gets its boundary from below.

Example 2.9. The descending manifolds of a Morse function form a canonical right flow module on $\mathcal{M}_f$. Such a canonical module does not exist in Floer theory.

Definition 2.10. A *bordism* between right flow modules $\mathcal{N}_0, \mathcal{N}_1$ over a flow category $\mathcal{M}$ is a map $\mathcal{W}\colon \mathrm{obj}(\mathcal{M}) \to \mathrm{Man}_{\mathrm{or}}$ with embeddings $\mathcal{W}(x) \times \mathcal{M}(x, y) \to \mathcal{W}(y)$ and $\mathcal{N}_i(x) \hookrightarrow \mathcal{W}(x)$ for $i \in \{0, 1\}$ such that

- $\dim(\mathcal{W}(y)) = k + 1 + |y|$ for $y \in \mathrm{obj}(\mathcal{M})$
- $\partial\mathcal{W}(y) = \mathcal{N}_0(y) \sqcup \mathcal{N}_1(y) \cup \bigcup_{|x|<|y|} \mathcal{W}(x) \times \mathcal{M}(x, y)$

with the same conditions as in Definition 2.14

[2] The sign depends on your convention; k denotes the dimension of the flow module defined a few lines later.

Definition 2.11. We call two right flow modules *(right) bordant* if there is a bordism between them and denote the *right bordism group* of $\mathcal{M}$ by $\Omega_*(\mathcal{M})$, where the addition is given by the disjoint union.

The same construction can be applied to left flow modules.

Definition 2.12. A *left flow module* over a flow category $\mathcal{M}$ is a map $\mathcal{N}\colon \operatorname{obj}(\mathcal{M}) \to \mathrm{Man}_{\mathrm{or}}$ together with the data of smooth embeddings $\mathcal{M}(x,y) \times \mathcal{N}(y) \to \mathcal{N}(x)$ such that

$$\partial \mathcal{N}(x) = \bigcup_y \mathcal{M}(x,y) \times \mathcal{N}(y).$$[3]

We say that $\mathcal{N}$ has *dimension* k if $\dim(\mathcal{N}(x)) = k - |x|$ for $x \in \operatorname{obj}(\mathcal{M})$. Denote the set of left flow modules over $\mathcal{M}$ of dimension k by $\Theta^k\mathcal{M}$.

Slogan: $\mathcal{N}(x)$ gets its boundary from above.

Remark 2.13. In particular, $\mathcal{N}(x)$ is a closed manifold if $|x|$ is maximal.

Definition 2.14. A *(left) flow bordism* between left flow modules $\mathcal{N}_0, \mathcal{N}_1 \in \Theta^k\mathcal{M}$ is an assignment $\mathcal{W}\colon \operatorname{obj}(\mathcal{M}) \to \mathrm{Man}_{\mathrm{or}}$ with embeddings $\mathcal{M}(x,y) \times \mathcal{W}(y) \hookrightarrow \mathcal{W}(x)$ and $\mathcal{N}_i(x) \hookrightarrow \mathcal{W}(x)$ for $i \in \{0,1\}$ such that

- $\dim(\mathcal{W}(x)) = k + 1 - |x|$ for $x \in \operatorname{obj}(\mathcal{M})$
- $\partial\mathcal{W}(x) = \mathcal{N}_0(x) \sqcup \mathcal{N}_1(x) \cup \bigcup_y \mathcal{M}(x,y) \times \mathcal{W}(y)$

where all these strata are of codimension 1 and intersect in codimension 2 strata.
The orientation of $\mathcal{N}_0(x)$ agrees with the orientation of the corresponding boundary stratum, while the orientation of $\mathcal{N}_1(x)$ differs by -1.

Definition 2.15. We call two left flow modules *(left) bordant* if there is a flow bordism between them and denote the *left bordism group* of $\mathcal{M}$ by $\Omega^\bullet(\mathcal{M})$.

Lemma 2.16. *Being (left) bordant is an equivalence relation.*

Proof. For reflexivity, let $\mathcal{W}(x) := \mathcal{N}(x) \times [0,1]$ for $x \in \operatorname{obj}(\mathcal{M})$. The relation is symmetric since we can simply take the same bordism with the opposite orientation. Transitivity follows from Proposition 1.17. □

Example 2.17. Left and right flow modules agree on a discrete flow category (i.e., no morphisms) and are simply maps from the set of objects to the class of (smooth oriented) closed manifolds.

Example 2.18. Let $\mathcal{M}$ be a flow module and fix $z \in \operatorname{obj}(\mathcal{M})$. Then $\mathcal{M}(z,\cdot)$ is a right flow module of dimension $|-z| - 1$ and $\mathcal{M}(\cdot, z)$ is a left flow module of dimension $|z| - 1$.

Remark 2.19. The goal is to build a space/spectrum from $\Omega_*\mathcal{M}$. However, this was not addressed in these lectures.

[3] If we want to keep track of orientations, it is $\partial\mathcal{N}(x) = \bigcup_y)(-1)^{something}\mathcal{M}(x,y) \times \mathcal{N}(y)$.

2.3 Computing the bordism groups

Note that if all objects of $\mathcal{M}$ have the same degree i, then the category is discrete. In particular, any left or right module is just a map $\mathcal{N}\colon \operatorname{obj}(\mathcal{M}) \to \mathrm{Man}_{\mathrm{or}}$ with all elements having the same dimension. In particular, we obtain an isomorphism

$$\Omega_k(\mathcal{M}) \to \bigoplus_{x \in \operatorname{obj}(\mathcal{M})} \Omega_{k+|x|}.$$

To use this observation for general flow categories, filter $\mathcal{M}$ according to the degree, that is, let $F^i\mathcal{M} \subset \mathcal{M}$ be the full subcategory with

$$\operatorname{obj}(F^i\mathcal{M}) := \{x \in \operatorname{obj}(\mathcal{M}) : |x| > i\}$$

Then we have canonical inclusions $F^i\mathcal{M} \hookrightarrow F^{i-1}\mathcal{M}$.

Proposition 2.20. *There exists an exact triangle*

$$\begin{array}{ccc} \Omega_k(F^i\mathcal{M}) & \longrightarrow & \Omega_k(F^{i-1}\mathcal{M}) \\ & {\scriptstyle -1}\nwarrow \quad \swarrow & \\ & \Omega_k(F^{i-1}\mathcal{M} \setminus F^i\mathcal{M}) & \end{array}$$

where the first map is given by extension by $\emptyset$, the second map is given by restriction, and the third map is induced by the map which sends $(N(x))_{|x|=i}$ to $\mathcal{N}\colon \operatorname{obj}(F^i\mathcal{M}) \to \mathrm{Man}_{\mathrm{or}}$ where

$$\mathcal{N}(y) = \bigsqcup_{|x|=i} N(x) \times \mathcal{M}(x,y). \tag{2.3.1}$$

Proof. Let us first note that the second map is well-defined by Remark 2.13 and that the second map is well-defined by the fact that $\dim(N(y) \times \mathcal{M}(y,x)) = k + |y| + |x| - |y| - 1 = k - 1 + |x|$ and the definition of a flow category. The fact that these maps respect the bordism relation follows from the definitions.
It is immediate that the composite $\Omega_k(F^i\mathcal{M}) \to \Omega_k(F^{i-1}\mathcal{M} \setminus F^i\mathcal{M})$ is zero, so suppose $[\mathcal{N}]$ is mapped to zero under $\Omega_k(F^{i-1}\mathcal{M}) \to \Omega_k(F^{i-1}\mathcal{M} \setminus F^i\mathcal{M})$. Then $N(x) = \partial W(x)$ for some manifold $W(x)$ of dimension $k+1$. Thus, we can define $\mathcal{W}\colon \operatorname{obj}(F^{i-1}\mathcal{M}) \to \mathrm{Man}_{\mathrm{or}}$ inductively by setting $\mathcal{W}(x) = W(x)$ for $|x| = i$. For $|y| = i+1$ define

$$\tilde{\mathcal{N}}(y) := \Big(\bigcup_{|x|=i} \mathcal{W}(x) \times \mathcal{M}(x,y)\Big) \cup_\partial \mathcal{N}(y)$$

where $\cup_\partial$ denotes the (oriented) connected sum along the boundary of the respective manifolds, using some collar neighbourhood. Then $\tilde{\mathcal{N}}(y)$ is a smooth oriented closed manifold of dimension $k + |y|$ and we can define $\mathcal{W}(y) := \tilde{\mathcal{N}}(y) \times [0,1]$. Let $\mathcal{N}(y) \hookrightarrow \mathcal{W}(y)$ be given by the inclusion into $\tilde{\mathcal{N}}(y) \times \{0\}$ and let $\tilde{\mathcal{N}}(y) \hookrightarrow \mathcal{W}(y)$ be the inclusion of the 1-boundary component.

Similarly we include $\mathcal{W}(x) \times \mathcal{M}(x,y)$ into $\tilde{\mathcal{N}}(y) \times s\{0\} \subset \mathcal{W}(y)$. Now proceed in this way. As $\mathcal{M}$ has finitely many objects, we obtain that $\mathcal{W}$ is a cobordism from $\mathcal{N}$ to the extension of $\tilde{\mathcal{N}}$ by $\emptyset$.
If $([N(x)])_{|x|=i} \in \Omega_k(F^{i-1}\mathcal{M} \backslash F^i\mathcal{M})$ maps to 0 in $\Omega_{k-1}(F^i\mathcal{M})$, then there exists $\mathcal{W}\colon \operatorname{obj}(F^i\mathcal{M}) \to \mathrm{Man}_{\mathrm{or}}$ such that $\dim(\mathcal{W}(x)) = k + |x|$ and

$$\partial\mathcal{W}(x) = \bigcup_{|z|<|x|} \mathcal{W}(z) \times \mathcal{M}(z,x) \cup \bigsqcup_{|y|=i} N(y) \times \mathcal{M}(y,x)$$

for $|x| > i$. Note that for $|x| = i+1$ we have $\partial\mathcal{W}(x) = \bigsqcup_{|y|=i} N(y) \times \mathcal{M}(y,x)$ and that thus for $|x| > i+1$ we have

$$\partial\mathcal{W}(x) = \bigcup_{|z|<|x|} \mathcal{W}(z) \times \mathcal{M}(z,x) \subset \mathcal{W}(x).$$

Define $\mathcal{N}'\colon \operatorname{obj}(F^{i-1}\mathcal{M}) \to \mathrm{Man}_{\mathrm{or}}$ by

$$\mathcal{N}(x) = \begin{cases} N(x) & |x| = i \\ \mathcal{W}(x) & |x| > i \end{cases} \tag{2.3.2}$$

and endow it with the structural maps of $\mathcal{W}$ (including the embeddings $N(x) \hookrightarrow \mathcal{W}(c)$). Then $[\mathcal{N}] \in \Omega_k(F^{i-1}\mathcal{M})$ and it maps to $([N(x)])_{|x|=i}$. Exactness at $\Omega_k(F^{i-1}\mathcal{M} \setminus F^i\mathcal{M})$ follows.
Finally, suppose the extension by $\emptyset$ of $\mathcal{N} \in \Theta_k(F^i\mathcal{M})$ is bordant to the empty flow module via $\mathcal{W}$. Then $\partial\mathcal{W}(x) = \emptyset$ for $|x| = i$, so we can define $[N] := ([\mathcal{W}(x)])_{|x|=i} \in \Omega_{k+1}(F^{i-1}\mathcal{M} \setminus F^i\mathcal{M})$ for $|x| = i$. Let $\tilde{\mathcal{N}}$ be defined by (2.3.1) using the $\mathcal{W}(x)$. Then the restriction of $\mathcal{W}$ to $\operatorname{obj}(F^i\mathcal{M})$ induces a bordism from $\mathcal{N}$ to $\tilde{\mathcal{N}}$. This completes the proof. □

Corollary 2.21. *There exists an exact triangle*

$$\begin{array}{ccc} \Omega^k(F^{i-1}\mathcal{M} \setminus F^i\mathcal{M}) & \longrightarrow & \Omega^k(F^{i-1}\mathcal{M}) \\ & {\scriptstyle +1}\nwarrow \quad \swarrow & \\ & \Omega^k(F^i\mathcal{M}) & \end{array}$$

where the first map is given by extension by $\emptyset$, the second map is given by restriction and the third map is induced by suitable surgery.

Example 2.22 (Bordism theory of Morse flow categories). Suppose $f\colon X \to \mathbb{R}$ is a Morse function on a closed smooth oriented manifold X. Given a smooth function $\pi\colon N \to X$ from a closed manifold N, we can define a right flow module $\mathcal{N}$ of $\mathcal{M}_f$ by setting

$$\mathcal{N}(x) := N \times_X \mathcal{D}(p)$$

for $p \in \mathrm{Crit}(f)$, where we take the fibre product over π and the evaluation map $\mathrm{ev}_0\colon \mathcal{D}(p) \to X$ at 0. We may assume without loss of generality that π is smooth by [BT13, Proposition 17.3] and that $\pi \pitchfork \mathrm{ev}_0$ as we can homotope it smoothly to such a map if necessary.

Proposition 2.23. *The resulting map $\Omega_k(X) \to \Omega_k(\mathcal{M}_f)$ is an isomorphism. In particular, the bordism theory of $\mathcal{M}_f$ is independent of the choice of Morse function f.*

3 The category of flow categories

The clategory $\mathcal{F}low$ of flow categories has as objects flow categories. The morphism space $\mathcal{F}low(\mathcal{M}_0, \mathcal{M}_1)$ consists of flow bimodules between the two flow categories. It will be a symmetric monoidal ($(\infty, 1)$, spectral, ...) category, which is linear over Ω_*.

Definition 3.1. Suppose $\mathcal{M}_0$ and $\mathcal{M}_1$ are flow categories. A $(\mathcal{M}_0, \mathcal{M}_1)$-*flow bimodule* of dimension k $\mathcal{P}$ is a map $\mathcal{P}\colon \operatorname{obj}(\mathcal{M}_0) \times \operatorname{obj}(\mathcal{M}_1) \to \mathrm{Man}_{\mathrm{or}}$ together with smooth embeddings

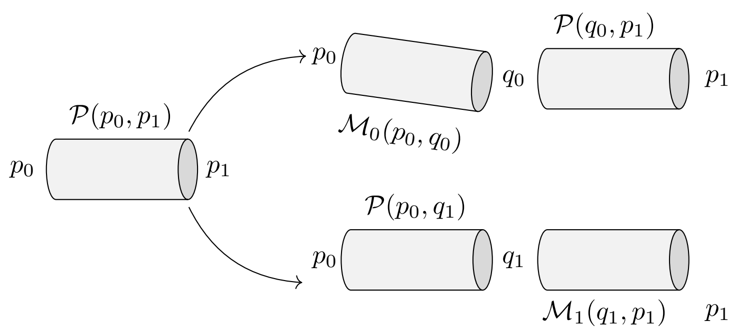
$$\mathcal{M}_0(x_0, y_0) \times \mathcal{P}(y_0, y_1) \times \mathcal{M}_1(y_1, x_1) \to \mathcal{P}(x_0, x_1)$$

into the boundary of codimension 1 such that

- $\dim(\mathcal{P}(x, y)) = k + |y| - |x| - 1$
- the boundary is given by

$$\partial\mathcal{P}(x_0, x_1) = \bigcup_{y_0, y_1} \mathcal{M}_0(x_0, y_0) \times \mathcal{P}(y_0, y_1) \times \mathcal{M}_1(y_1, x_1).$$

- The codimension $\ell > 1$ strata are the images of iterated composition maps.



Example 3.2. In Hamiltonian Floer theory, we have canonical continuation maps for different choices of Hamiltonians.

To put this into context, one can consider the following' extension of the category of rings'. This was shown to me by Hiro Lee Tanaka.

Example 3.3 (Bimodules over rings). Let Ring be the 2-category whose objects are rings, and where for rings R and S rings the space of 1-morphisms is given by ${}_R\mathrm{Mod}_S$. The usual category of rings includes into Ring as a ring morphism $R \to S$ defines a (R, S)-bimodule structure on S.

However, we are only interested in flow bimodules up to bordism.

Definition 3.4. Suppose $\mathcal{M}_0$ and $\mathcal{M}_1$ are flow categories and $\mathcal{P}_0, \mathcal{P}_1$ are two flow bimodules between them. A *flow bimodule bordism* from $\mathcal{P}_0$ to $\mathcal{P}_1$ is a map

$$\mathcal{W}\colon \operatorname{obj}(\mathcal{M}_0) \times \operatorname{obj}(\mathcal{M}_1) \to \mathrm{Man}_{\mathrm{or}}$$

together with smooth embeddings

$$\mathcal{M}_0(x_0, y_0) \times \mathcal{P}(y_0, y_1) \times \mathcal{M}_1(y_1, x_1) \to \mathcal{P}(x_0, x_1)$$

and $\mathcal{P}_i(x_0, x_1) \hookrightarrow \mathcal{W}(x_0, x_1)$ into the boundary of codimension 1 such that

- $\dim(\mathcal{W}(x_0, x_1)) = k + 1 + |x_1| - |x_0| - 1$
- the boundary is given by

$$\partial\mathcal{W}(x_0, x_1) = \mathcal{P}_0(x_0, x_1) \sqcup \mathcal{P}_1(x_0, x_1) \cup \bigcup_{y_0, y_1} \mathcal{M}_0(x_0, y_0) \times \mathcal{P}(y_0, y_1) \times \mathcal{M}_1(y_1, x_1)$$

  with the embedding of $\mathcal{P}_0(x_0, x_1)$ being orientation-preserving and the embedding of $\mathcal{P}_1(x_0, x_1)$ being orientation-reversing.

Definition 3.5. We define $\mathcal{F}low(\mathcal{M}_0, \mathcal{M}_1)$ to be the set of bimodules modulo the relation of being bordant.

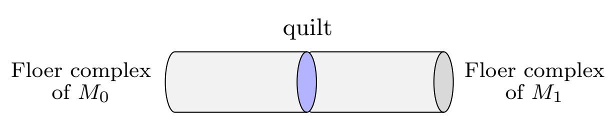
This admits an Ω_*-module structure given by $[N] \cdot [\mathcal{P}] := [N \times \mathcal{P}]$.

Example 3.6. Given a flow category $\mathcal{M}$ we have

$$\mathcal{F}low(\mathcal{M}, *) = \Omega^\bullet(\mathcal{M}) \qquad \mathcal{F}low(*, \mathcal{M}) = \Omega_*(\mathcal{M}).$$

Flow bimodules appear naturally in (symplectic) geometry.

Example 3.7. One motivation is given by the notion of quilts as introduced in [WW09]



where a Lagrangian correspondence $L \subset M_0 \times M_1$ is used to define a functor between their Floer complexes.

To motivate the formula for the composition of flow bimodules, let us return to Example 3.3. If $M \in {}_R\mathrm{Mod}_S$ and $N \in {}_S\mathrm{Mod}_T$, then their composition $M \circ N \in {}_S\mathrm{Mod}_T$ is given by $M \otimes_S N$ with the obivious bimodule structure. We will do the analogous constructions, which is somewhat more complicated as we have to preserve topologies and smooth structures.

Definition 3.8. Suppose $\mathcal{P}_{01}$ is a $(\mathcal{M}_0, \mathcal{M}_1)$-bimodule and $\mathcal{P}_1$ a $(\mathcal{M}_0, \mathcal{M}_1)$-bimodule. The *composite bimodule* is $\mathcal{P}_{02} := \mathcal{P}_{01} \circ \mathcal{P}_{12}$ is the $(\mathcal{M}_0, \mathcal{M}_2)$-bimodule with

$$\mathcal{P}_{02}(x_0, x_2) = \mathrm{coeq}\Big(\bigsqcup_{x_1, y_1} \mathcal{P}_{01}(x_0, x_1) \times \mathcal{M}_1(x_1, y_1) \times \mathcal{P}_{12}(y_1, x_2) \rightrightarrows \bigsqcup_{x_1} \mathcal{P}_{01}(x_0, x_1) \times \mathcal{P}_{12}(x_1, x_2) \Big) \tag{3.0.1}$$

with the induced structural maps.

By Proposition 1.17, there exists a smooth structure on $\mathcal{P}_{02}(x_0, x_2)$ which is unique up to a contractible choice of ambient isotopy. Endowed with this smooth structure, all structural maps are smooth embeddings. Hence, the composition is well-defined.

Lemma 3.9. *The composition of flow bimodules is associative.*

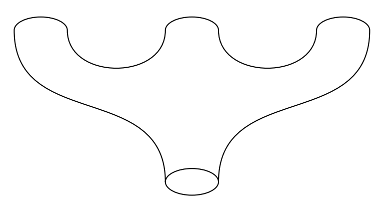
Proof. It suffices to show that both $\mathcal{P}_{01} \circ \mathcal{P}_2$ and $\mathcal{P}_0 \circ \mathcal{P}_{12}$ are both the colimit of

$$\begin{array}{c} \displaystyle\bigsqcup_{p_1, q_1, p_2, q_2} \mathcal{P}_0(p_0, p_1) \times \mathcal{M}_1(p_1, q_1) \times \mathcal{P}_1(q_1, q_2) \times \mathcal{M}_2(q_2, p_2) \times \mathcal{P}_2(p_2, p_3) \\ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\ \displaystyle\bigsqcup_{p_1, p_2} \mathcal{P}_0(p_0, p_1) \times \mathcal{P}_1(p_1, p_2) \times \mathcal{P}_2(p_2, p_3) \end{array}$$

This is left as an exercise. $\square$

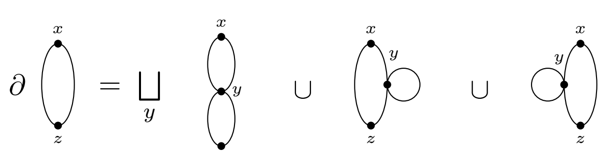
4 Further directions

There are three natural next questions:



1. Multiplicativity[4]: We don't want to take products of flow categories because that is annoying. Instead, we consider bimodules with multiple inputs (but only one output).

2. The fact that the Lagrangian Floer complex is a curved complex (i.e., the differential does not square to 0) in general:



[4]That is, the symmetric monoidal structure mentioned above.

Thus, we have to study the analogue of an A_∞-algebra (Fukaya category) using the notion of a *flow multicategory*. One can then show that flow multicategories form a symmetric monoidal 2-category (that is, if $\mathcal{M}_0$ and $\mathcal{M}_1$ are flow categories, then $\mathcal{F}low_{\mathcal{M}_0,\mathcal{M}_1}$ is a category).
Note: We have seen in the previous lecture the case of $\mathcal{M}_0 = \mathcal{M}_1 = \{\mathrm{pt}\}$. The moduli spaces appearing in the Floer theory of two Lagrangians are the prototypical examples of $\mathcal{F}low_{\mathrm{pt},\mathrm{pt}}$.

3. technical issues: gluing and bubbling

We will only discuss the last point in this lecture.

4.1 Gluing

We might have a topological manifold $\mathcal{M}(x, z)$ with corners but it may not admit any smooth structure near the boundary strata $\mathcal{M}(x, y) \times \mathcal{M}(y, z)$. One might try to build collars, but these often rely on representatives (the manifolds $\mathcal{M}(x, y)$ often consist of equivalence classes of geometric objects) and thus are highly local.

Problem: If we construct a collar, we do not know whether it is smooth.

Solution: Work with topological manifolds with structure group $O(n)$.

Definition 4.1. A topological manifold M^n is said to *have structure group* $O(n)$ if the classifying map $M \to \mathrm{Homeo}(\mathbb{R}^n, 0)$ of its tangent microbundle $T_\mu M$ admits a lift to $O(n)$.

We refer to [AMS21] for a deeper discussion of this structure and its consequences. Fortunately, manifolds with this structure were studied in the later half of the 20th century; see the relevant references in [AMS21]. In summary, we have the following result.

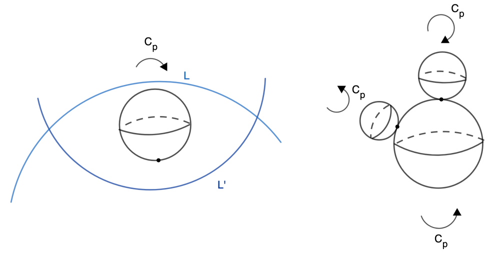
Theorem 4.2 (Smoothing theory)**.** *The inclusion map* $\Omega^{\mathrm{Diff}} \to \Omega^{\mathrm{Top},O}$ *is an isomorphism.*

This solves the issues with gluing, so we now turn to the more serious issue of bubbling.

4.2 Bubbling

Sphere bubbling is worse than disc bubbling as spheres (with one marked point) have a more complicated symmetric group than (once-marked) disc whose symmetry group is contractible and does not have finite subgroups. Thus, in the case of sphere bubbling, one cannot deform the

Floer equation to achieve transversality without breaking symmetry (except perhaps in dimension 4). Hence, one has to deal with spaces that are neither manifolds nor orbifolds. These spaces carry various names, such as *Kuranishi spaces* or *derived orbifolds*. The basic idea is *finite-dimensional reduction*: locally model your moduli spaces as the zero set of the section of a finite-dimensional vector bundle over a topological/smooth manifold. More explicitly, given $\mathcal{M}$ we want to find an open cover $\{U_i\}_{i\in I}$, finite groups Γ_i and Γ_i-manifolds $\mathcal{T}_i$ and Γ_i-representations V_i together with an equivariant function $s_i\colon \mathcal{T}_i \to V_i$ such that



$$s_i^{-1}(0)/\Gamma_i \cong U_i.$$

See [MW17] and [Par16, Chapter 9] for such a construction in the case of closed Gromov-Witten theory and [Par16, Chapter 10] for Hamiltonian Floer theory.

Question: How to assemble this globally?

See [FO99, MW17, Par16] for three approaches. Unfortunately, any of these constructions is highly technical and has a large notational overhead arising from the difficulty of going from local to global information. Thus, one might hope for global Kuranishi charts, which make this passage superfluous.

Slogan: The moduli spaces appearing in Floer theory have natural geometric global Kuranishi charts.

Definition 4.3. Let $\mathcal{M}$ be a (compact Hausdorff) space. A *global Kuranishi chart* $\mathcal{K}$ is a tuple $(G, \mathcal{T}, \mathcal{E}, \mathfrak{s}, \psi)$ where G is a compact Lie group acting almost freely and locally linearly[5] on the topological manifold $\mathcal{T}$ and $\mathcal{E} \to \mathcal{T}$ is a G-vector bundle with $\mathfrak{s}\colon \mathcal{E} \to \mathcal{T}$ a G-equivariant section. Finally, $\psi\colon \mathfrak{s}^{-1}(0)/G \to \mathcal{M}$ is a homeomorphism.
The *virtual dimension* of $\mathcal{M}$ (with respect to $\mathcal{K}$) is $\mathrm{vdim}_{\mathcal{K}}(\mathcal{M}) = \dim(\mathcal{T}) - \mathrm{rank}(\mathcal{E}) - \dim(G)$.

Definition 4.4. An *equivalence* $\mathcal{K} \to \mathcal{K}'$ of two global Kuranishi charts for $\mathcal{M}$ consists of the following data

- a homomorpghism $\alpha\colon G \to G'$
- an α-equivariant map $\phi\colon \mathcal{T} \to \mathcal{T}'$ so that $\ker(\alpha)$ acts freely on the fibres of ϕ

[5]That is, all stabilisers are finite, and for each $x \in \mathcal{T}$ we can find a neighbourhood $U \subset \mathcal{T}$ such that U is homeomorphic to an open set in $V \times_{G_x} G$, where V is a G_x-representation. In particular, $\mathcal{T}/G$ is a topological orbifold

- an α-equviariant vector bundle map $\Phi\colon \mathcal{E} \to \phi^*\mathcal{E}'$ such that $\Phi\mathfrak{s} = \phi^*\mathfrak{s}'$.

so that

(i) the induced map $\bar{\phi}\colon \mathfrak{s} \in (0)/G \to (\mathfrak{s}')^{-1}(0)/G'$ is a homeomorphism and $\psi'\bar{\phi} = \psi$

(ii) the following square is cartesian

$$\begin{array}{ccc} \mathcal{E} \times_{G'} G & \longrightarrow & \mathcal{E}' \\ \downarrow & & \downarrow \\ \mathcal{T} \times_{G'} G & \longrightarrow & \mathcal{T}' \end{array}$$

Definition 4.5. We call two global charts *equivalent* if they are related by a zigzag of equivalences as in Definition 4.4.

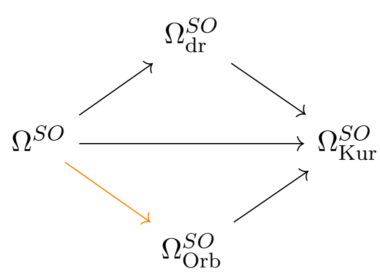
Definition 4.6. A *bordism between global Kuranishi charts* $\mathcal{K}_0$ and $\mathcal{K}_1$ consists of a Kuranishi chart $\widetilde{\mathcal{K}}$ with boundary such that $\partial\widetilde{\mathcal{K}} = \mathcal{K}_0' \sqcup \mathcal{K}_1'$ with $\mathcal{K}_i'$ equivalent to $\mathcal{K}_i$ for $i \in \{0, 1\}$.

In [AMS21], Abouzaid-McLean-Smith constructed such a global Kuranishi chart for the moduli space of stable maps of genus zero.

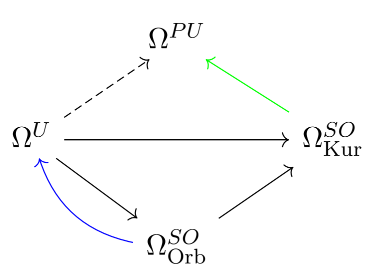
Theorem 4.7. *Let (X, ω) be a closed symplectic manifold, J an ω-tame almost complex structure and $\beta \in H_2(X; \mathbb{Z})$. Then $\overline{\mathcal{M}}^J_{0,n}(X, \beta)$ admits a global Kuranishi chart.*

Remark 4.8. Kur is like an equivariant version of derived manifolds.

Thus, we can define a Kuranishi bordism ring $\Omega_{\mathrm{Kur},*}$ where $*$ indicates the virtual dimension. We have commutative diagrams (which have variants for Spin, etc.)



where the noninjectivity of the map indicated in orange was shown by [Ang10] and



where the existence of the blue and green arrow is conjectural so far. The blue arrow corresponds to a splitting, while the green arrow would be a 'periodic splitting'.
These conjectures would imply that one can define the Fukaya category over complex cobordism.

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