Proof of the main theorem, part I

We want to show that Flow admits a stable structure. To do so, we make use of the following result.

Prop. [Lurie, '12]
Let $\mathcal{C}$ be an ∞-cat. Then, $\mathcal{C}$ is stable iff the following conditions are met:
i) $\mathcal{C}$ is pointed;
ii) $\forall x \in \mathrm{Ob}(\mathcal{C})$, the pushout

$$\begin{tikzcd} x \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & \Sigma x \end{tikzcd}$$

exists and defines an auto-equivalence $\Sigma : \mathcal{C} \to \mathcal{C}$;
iii) every morphism $f: X \to Y$ admits a cofiber.

<u>This talk</u>: i) & the first part of ii).

I. Preliminary facts on semisimplicial sets

The functor $i: \Delta_+ \to \Delta$ induces a forgetful functor

$$\underbrace{\mathrm{Fun}(\Delta^{op}, \mathrm{Set})}_{\mathrm{sSet}} \xrightarrow{i^*} \underbrace{\mathrm{Fun}(\Delta_+^{op}, \mathrm{Set})}_{\mathrm{ssSet}}.$$

Through abstract nonsense, i^* admits a left adjoint $i_!$, i.e.

$$\mathrm{sSet}(i_!(-), -) \underset{nat.}{\cong} \mathrm{ssSet}(-, i^*(-)).$$

Concretely, $i_!$ is obtained by adjoining degeneracies so that if $\sigma: [v_0 : \ldots : v_n] \to |X|$ is a simplex of the geometric realization of X, then

$$[v_0 : \ldots : v_i : v_i : \ldots : v_j : v_j : \ldots : v_n] \longrightarrow |i_! X| = |X|,$$

extended trivially, is a simplex of the geometric realization of $i_! X$.

Therefore, we have that

$$i_! \Delta_+^n = \Delta^n, \quad i_!(\partial \Delta_+^n) = \partial \Delta^n, \quad i_!(\Lambda^n_{k,+}) \cong \Lambda^n_k.$$

Furthermore, if X is a semisimplicial set with maps $s_0, \ldots, s_n : X_n \to X_{n+1}$ (as constructed in §6), then the resulting simplicial set $\tilde{X}$ has the property that

$$i^* \tilde{X} = X$$

$$\rightsquigarrow \quad i^* \mathrm{Fun}(\Delta^n, \tilde{X}) \cong \mathrm{Fun}(\Delta_+^n, X)$$
$$i^* \mathrm{Fun}(\partial \Delta^n, \tilde{X}) \cong \mathrm{Fun}(\partial \Delta_+^n, X)$$

$$\rightsquigarrow \quad \pi_n(\tilde{X}, x) = \pi_n(X, x) \quad \text{if } \tilde{X} \text{ is a Kan complex.}$$

Moral: Up to homotopy, we can simply work with the semi-simplicial set Flow — everything for the simplicial set will follow automatically.

2. Pointed mapping spaces in Flow

The space $\mathrm{Flow}(\mathbb{X}, \mathbb{Y})$ of morphisms $\mathbb{X} \to \mathbb{Y}$ is given by the pullback diagram (in sSet)

$$\begin{tikzcd}
\mathrm{Flow}(\mathbb{X},\mathbb{Y}) \arrow[r] \arrow[dd] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Fun}(\Delta^1, \mathrm{Flow}) \arrow[d] \\
 & \mathrm{Fun}(\partial\Delta^1, \mathrm{Flow}) \arrow[d, "\cong"] \\
\Delta^0 \arrow[r, "\{\mathbb{X}\}\times\{\mathbb{Y}\}"] & \mathrm{Flow} \times \mathrm{Flow}.
\end{tikzcd}$$

Concretely, this means that the n-simplices of $\mathrm{Flow}(\mathbb{X}, \mathbb{Y})$ are given by morphisms

$$\Delta^1 \times \Delta^n \longrightarrow \mathrm{Flow}$$

s.t. their restriction to $\partial\Delta^1 \times \Delta^n$ is $\{\mathbb{X}, \mathbb{Y}\}$.

Note that $\mathrm{Flow}(\mathbb{X}, \mathbb{Y})$ has a favored point in $\emptyset_{\mathbb{X},\mathbb{Y}}$, the empty 1-simplex, given as follows. If $\mathrm{Ob}(\mathbb{X}) = \mathcal{P}_0$ and $\mathrm{Ob}(\mathbb{Y}) = \mathcal{P}_1$, then $\emptyset_{\mathbb{X},\mathbb{Y}}$ has object given by $\mathcal{P} = (\mathcal{P}_0, \mathcal{P}_1)$, and

$$\emptyset_{\mathbb{X},\mathbb{Y}}(p,q) := \begin{cases} \mathbb{X}(p,q) & \text{if } p,q \in \mathrm{Ob}(\mathbb{X}) \\ \mathbb{Y}(p,q) & \text{if } p,q \in \mathrm{Ob}(\mathbb{Y}) \\ \emptyset & \text{otherwise.} \end{cases}$$

Lemma: The ∞-cat. Flow is pointed with zero object $\emptyset$, the empty flow cat. ($\mathrm{Ob}(\emptyset) = \emptyset$).

Proof: We need to prove that the spaces $\mathrm{Flow}(\emptyset, \mathbb{X})$ and $\mathrm{Flow}(\mathbb{X}, \emptyset)$ are contractible $\forall \mathbb{X} \in \mathrm{Ob}(\mathrm{Flow})$, i.e.

$$\pi_n(\mathrm{Flow}(\emptyset, \mathbb{X}), \emptyset_{\emptyset\mathbb{X}}) = \pi_n(\mathrm{Flow}(\mathbb{X}, \emptyset), \emptyset_{\mathbb{X}\emptyset}) = 0 \quad \forall n.$$

For this, it suffices to prove that whenever we have $\partial\Delta^n \longrightarrow \mathrm{Flow}$ s.t. one vertex is sent to $\emptyset$, there is an extension $\Delta^n \longrightarrow \mathrm{Flow}$.

However, this is clear: it suffices to associate to the interior of Δ^n the empty n-simplex.

∎

3. Flow is a semistable quasicategory (part 2)

We need to show that there is a pair of endofunctors (Σ, Σ^{-1}) of Flow which induces natural equivalences

$$\underset{\substack{\cong \\ \mathrm{Flow}(\Sigma^{-1}\mathbb{X},\, \Sigma^{-1}\mathbb{Y})}}{\Omega\mathrm{Flow}(\Sigma^{-1}\mathbb{X}, \mathbb{Y})} \longleftarrow \mathrm{Flow}(\mathbb{X}, \mathbb{Y}) \longrightarrow \underset{\substack{\cong \\ \mathrm{Flow}(\Sigma\mathbb{X},\, \Sigma\mathbb{Y})}}{\Omega\mathrm{Flow}(\mathbb{X}, \Sigma\mathbb{Y})},$$

where
$$\Omega Flow(\mathbb{X},\mathbb{Y}) := Fun\left(\Delta^1/\partial\Delta^1,\ Flow(\mathbb{X},\mathbb{Y})/\emptyset_{\mathbb{X},\mathbb{Y}}\right).$$
On the level of objects, we set $\Sigma = \Sigma^{-1} = 1\!\!1$ (this will become clear in the next talk), and we only have one equivalence
$$Flow(\mathbb{X},\mathbb{Y}) \longrightarrow \Omega Flow(\mathbb{X},\mathbb{Y}).$$
$$\overset{\text{II}}{\xrightarrow{\sim}} \quad \Sigma Flow(\mathbb{X},\mathbb{Y}) \longrightarrow Flow(\mathbb{X},\mathbb{Y})$$
But a morphism $\Sigma X \to Y$ (where Y is pointed) is specified by a sequence $\{\sigma_n : X_n \to Y_{n+1}\}_{n\geq 0}$ which resp. the simplicial id. and s.t.
$$d_0\sigma_n = d_1 \text{ ---- } d_{n+1}\sigma_n = * \leftarrow$$ map that sends everything to the special point of Y.

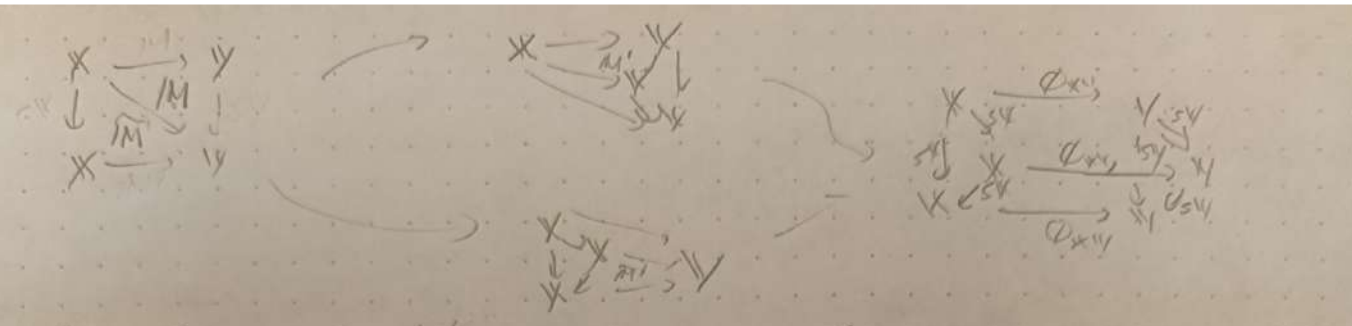
Lemma 7.8: Such maps exist.

__Proof:__ We need to construct
$$\sigma_n : Fun_{\mathbb{X},\mathbb{Y}}(\Delta^1_+\otimes\Delta^n_+, Flow) \longrightarrow Fun_{\mathbb{X},\mathbb{Y}}(\Delta^1_+\otimes\Delta^{n+1}_+, Flow)$$
$$\subset Fl_{\partial\Delta^1_+\otimes\Delta^n_+} = \{\mathbb{X},\mathbb{Y}\}$$
Let's start with $n=0$. Then, an element on the right is a square:
$$\begin{tikzcd} \mathbb{X} \arrow[r, "\emptyset_{\mathbb{X},\mathbb{Y}}"] \arrow[d, "s\mathbb{X}"'] \arrow[dr, phantom, "\mathbb{M}' \quad \emptyset_{\mathbb{X},\mathbb{Y}} \quad \mathbb{M}''"] & \mathbb{Y} \arrow[d, "s\mathbb{Y}"] \\ \mathbb{X} \arrow[r, "\emptyset_{\mathbb{X},\mathbb{Y}}"'] & \mathbb{Y} \end{tikzcd}$$
We decompose this square in 2 2-simplices and construct two maps $Fun(\Delta^1_+, Flow) \longrightarrow Fun(\Delta^2_+, Flow)$ which match along an edge.

We do as above, where $Ob(\mathbb{M}') = \underbrace{Ob(\mathbb{X})}_{P_0} \sqcup \underbrace{Ob(\mathbb{Y})}_{P_1} \sqcup \underbrace{Ob(\mathbb{Y})}_{P_2}$, and
$$\mathbb{M}'(p,q) := \begin{cases} \mathbb{M}(p,q) & \text{if } p\in P_0,\ q\in P_2 \\ \emptyset & \text{if } p\in P_0,\ q\in P_1 \\ s\mathbb{Y}(p,q) & \text{if } p\in P_1,\ q\in P_2 \\ \mathbb{X}(p,q) & \text{if } p,q\in P_0 \\ \mathbb{Y}(p,q) & \text{if } p,q\in P_1 \text{ or } p,q\in P_2 \end{cases}$$
$\mathbb{M}''$ is defined similarly, but $P_1 = Ob(\mathbb{X})$ instead.

For general n, it suffices to divide $\Delta^1_+\otimes\Delta^n_+$ in $n+1$ $(n+1)$-simplices and $\Delta^1_+\otimes\Delta^{n+1}_+$ in $n+2$ $(n+2)$-simplices and use a similar trick. For example:

$(n=1)$



We get the identities $d_0\sigma_n = d_1 \cdots d_n\sigma_n = x$ from our relabeling, which associate to lower dim. faces empty simplices