Floer homology & smooth structures

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(joint work with Ivan Smith)

$(X^{2n}, \omega = d\lambda)$ exact symp mfld.

Q: closed, exact Lag in X?

* **Ex:** (Gromov) $\nexists$ closed exact Lag in $\mathbb{C}^n$.

* **Ex:** $Q \underset{0\text{-sect}^n}{\hookrightarrow} T^*Q$

Conjecturally, this is the only one up to isotopy.

* **Thm** (Abouzaid + Kragh) Any closed exact $L \subset T^*Q$ is (simple) htpy equiv to Q.

Some steps in proof:

Prove L & Q are isomorphic in $\mathcal{F}uk(T^*Q)$

[Recall: $End(L) \cong H^*(L)$
$\quad \wr\wr$
$End(Q) \cong H^*(Q)$]

* **Thm** (Kervaire - Milnor)
There are exactly 28 mflds htpy equiv to S^7
——— 2 ——— S^8
$\hookrightarrow$ but not diffeom.

Let $\{S^8, \Sigma\}$ be these mflds.
Q: Can we prove Σ can't be a Lag in T^*S^8?

* **Prop:** S^8, Σ are not framed cobordant
[Recall framed cob is W: $\partial W = S^8 \amalg \Sigma$, $TW \oplus \mathbb{R}^k \cong \mathbb{R}^{8+k}$)
but S^8 is framed cobordant to itself.

* **Thm:** Suppose $L, K \subseteq X$ are closed exact Lag. Assume they are homology spheres, + "stable Gauss map is 0", and $L \cong K$ in $\mathcal{F}uk(X)$. Then $[L]-[K]$ in Ω^{fr}_m is 2-torsion

* **Cor:** 1. Σ isn't a Lag in T^*S^8.
2. ——— $T^*S^8 \# T^*S^8$

* **Thm** (Large)
There is a "spectral Fukaya cat":
- Ob: Closed exact Lag. ~~...~~ with vanishing stable Gauss map
- Morphisms, $HF_*(L, K; \mathbb{S})$
modules over framed bordism ring Ω^{fr}_*.

Start with $L \cong K$ in $\mathcal{F}uk(X; \mathbb{Z})$ ← computable
incomputable, but more info!

* **Lemma:** If $L \cong K$ in $\mathcal{F}uk(X; \mathbb{S})$, then L & K are framed cob.

Q: How to bootstrap info from $\mathcal{F}uk(X; \mathbb{Z})$ to $\mathcal{F}uk(X; \mathbb{S})$?

Q: If $L \cong K$ in $\mathcal{F}uk(X; \mathbb{Z})$, does this imply $L \cong K$ in $\mathcal{F}uk(X; \mathbb{S})$?
NO.

The spectral Fuk cat has objects • Lag + extra data,
• Morphisms are the flow bordism gps $\Omega^{fr}_*(\mathcal{M}^{LK})$

$\mathcal{M}^{LK}$ is a framed flow cat
· Objects $L \pitchfork K$
· Morphisms $\mathcal{M}^{LK}(x, y) = \{x \ldots y\}$ (L, K)

(Thm Large, Porcelli)

There's a functor $\Phi: \mathcal{F}uk(X, \mathbb{S}) \to \mathcal{F}uk(X; \mathbb{Z})$

$$L \longmapsto L$$
$$HF_*(L, K; \mathbb{S}) \longrightarrow HF_*(L, K; \mathbb{Z})$$
$$\mathcal{A} \in \Omega_i(\mathcal{M}^{LK}) \longmapsto \sum_{x \in L \cap K} (\# \mathcal{A}(x)) x$$

$:= 0$ if $\dim \neq 0$.

(2 equiv flow modules are mapped to homologous chains)

* **Prop:** If $\alpha: L \to K$ is a morphism in $\mathcal{F}uk(X; \mathbb{S})$, st $\Phi(\alpha)$ is an $\cong$, then α was too (Φ is conservative).

Compare: If $f: E \to F$ is a map of (finite) spectra st $f_*: H_*(E) \to H_*(F)$ is an $\cong$, then f was too.

$\rightsquigarrow$ Lifting problem:

Φ: $\mathcal{F}uk(X; \mathbb{S}) \longrightarrow \mathcal{F}uk(X; \mathbb{Z})$

$(\alpha: L \to K) \in HF_*(L, K; \mathbb{Z})$

Can this come a map in $\mathcal{F}uk(X; \mathbb{S})$.

$\mathbb{Z} = \Omega^{fr}_{\leq 0}$, $\mathbb{S} = \Omega^{fr}_*$ (PT)

* **def:** $\mathcal{F}$ flow category (in theory framed but let's ignore).
A truncated $\tau_{\leq i}$ - flow module $\mathcal{A}$ over $\mathcal{F}$, is defined same as flow module, but only require $\mathcal{A}(x)$ when $\dim \mathcal{A}(x) = -|x| \leq i$.
and $\exists$ such mflds of $\dim = i+1$. (†)

A $\tau_{\leq 0}$ flow module over $\mathcal{F}$ consists of: compact 0-mflds $\mathcal{A}(x)$ $\forall |x| = 0$ $\rightsquigarrow \sum_{|x|=0} (\# \mathcal{A}(x)) \cdot x \in C_*(\mathcal{F})$.

(†) $\Rightarrow$ closed chain. So $[\mathcal{A}] \in H_*(\mathcal{F})$.

$$\begin{array}{ccc} HF_*(L, K, \mathbb{S}) & & \\ \| & & \\ \Omega_*(\mathcal{M}^{LK}) & \longrightarrow & \tau_{\leq m} \Omega_*(\mathcal{M}^{LK}) \\ & & \downarrow \\ & & \tau_{\leq m-1} \Omega_*(\mathcal{M}^{LK}) \\ & & \downarrow \\ & & \vdots \\ & & \downarrow \\ & & (\alpha: L \xrightarrow{\cong} K) \in HF_*(L, K; \mathbb{Z}) \end{array}$$

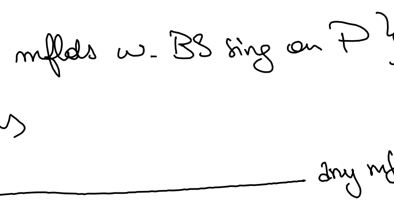
* **Prop:** A class $\alpha \in \tau_{\leq i} \Omega_*(\mathcal{F})$ lifts to $\tilde{\alpha} \in \tau_{\leq i+1} \Omega_*(\mathcal{F})$
$\Leftrightarrow$ some class $[\mathcal{O}] \in HF_{i+2}(L, K; \Omega_{i+1})$ vanishes.

3. FLOW MODULES & SINGULARITIES

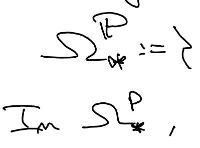
Let P closed mfld

* **def:** A mfld M with Baas - Sullivan singularities on P has singularities modelled $Cone(P) = P \times [0,1] / P \times \{1\}$
or (stuff) $\times Cone(P)$.

* **Ex:** $P = S^1$



sing modelled on S^1



$D^2 \times S^1$, collapse bdry $\quad D^2 \times S^1 / S^1 \times S^1$
sing modelled on T^2

* **def:** $\Omega^P_* := \{$closed mflds w. BS sing on $P\} /$ cobordism.

$\mathbb{P}$ set of manifolds
$\Omega^{\mathbb{P}}_* := \{$ ——— any mfld in $\mathbb{P}\} /$ cobordism

* **Ex.** In Ω^P_*, $[P] = 0$. Let $W = P \times [0,1] / P \times \{1\}$

* **Thm** (AB) $\mathcal{F}low^{fr} \simeq \mathbb{S}$- mod

Conj: $\mathcal{F}low^{fr, P} \simeq \mathbb{S}/p$ - mod

Noah Porcelli

Framed flow categories & spectra (§8 in AB)

Assume
- $\Gamma = 0$
- work with manifolds instead of dOrb
- things are framed

$\ast$ **Thm:** $\mathrm{Flow}^{fr} \simeq \mathrm{Spectra}$

(1.14) $\ast$ **Prop.** Let (X, ω) be a sympl. mfld st

no orb singularities — 1. X is aspherical $\pi_2 X = 0$ (sympl. asph w. $\pi_2(X) = 0$ is enough)

framings $\to$ 2. $TX \cong E \otimes \mathbb{C}$

$\leftarrow$ some vector bundle.

Then associated to a ham H on H, there is a framed flow cat.

$HF(H, \mathrm{Flow}^{fr})$. By Thm, get a spectrum.

(Produced earlier by Large, CJS. No proof that they coincide, but should)

<u>Next step</u>: Build $\mathcal{F}uk(X; \mathbb{S})$ (Fuk cat enriched in spectra)

1. Make Flow^{fr} into sym. mon. cat.
2. Build $\mathcal{F}uk(X; \mathrm{Flow}^{fr})$
3. Use $\mathrm{Flow}^{fr} \simeq \mathrm{Sp} \rightsquigarrow \mathcal{F}uk(X, \mathbb{S})$

[$\ast$ **Thm:** $\mathrm{Flow}^{fr} \simeq \mathrm{Spectra}$]

Question: What is Flow^{U}? (smooth mflds + complex structure)

$\to$ How to prove the theorem?

<u>Step 1</u> Flow^{fr} generated by flow cat w. single pt, $\star$.

$\ast$ <u>def</u>: $\mathcal{C}$ stable ∞-cat. is gen. by $X \in \mathcal{C}$ if the smallest subcategory containing X and all colimits is $\mathcal{C}$ (+all shifts).

($\widetilde{\mathrm{Flow}}$ means mapping spectrum, Flow is mapping space)

$\Rightarrow \mathrm{Flow}^{fr} \simeq R\text{-mod}$, $R = \widetilde{\mathrm{Flow}}^{fr}(\star, \star)$ (ring spectrum)

<u>Step 2</u> Prove $R \simeq \mathbb{S}$. $\to \mathrm{Flow}^{fr} \simeq \mathbb{S}\text{-mod} \simeq \mathrm{Sp}$.

Let's do step 1. (Sketch)

Pick $\mathbb{X}$ flow cat, show can be built by iterated cones (cofibres, which are colimits). Assume $\mathbb{X}$ has finitely many objects.

$\forall x \in \mathrm{Ob}(\mathbb{X})$, $\dim V_x =: |x| \in \mathbb{Z}$.

If $\mathbb{X}(x, y) \neq \emptyset$, $|x| > |y|$.

$\forall \alpha \in \mathbb{Z}$, define $\mathbb{X}_{[-\infty, \alpha]}$ is the flow cat w. objects x st $|x| \leq \alpha$ & same morphisms.

$\ast$ <u>Prop</u>: $\exists$ maps $\underbrace{\mathbb{X}_{[-\infty, \alpha+1]} \to \mathbb{X}_{[-\infty, \alpha]}}_{\text{(in Flow}^{fr})}$ which fit into a cofibre sequence:

$$\begin{tikzcd}
\mathbb{X}_{[-\infty, \alpha+1]} \arrow[d, "s(\text{everything in deg } \alpha+1)"'] & \mathbb{X}_{[-\infty, \alpha]} \arrow[l, "s\mathbb{X}"'] \\
\coprod\limits_{\substack{x \in X \\ |x| = \alpha+1}} \Sigma^{n+1} \star &
\end{tikzcd}$$

diagonal bimodule

$\mathbb{X}$ finite $\Rightarrow \exists\, \alpha, \beta$ st
- $\mathbb{X}_{[-\infty, \alpha]} = \emptyset$
- $\mathbb{X}_{[-\infty, \beta]} = \mathbb{X}$

Inductively show $\mathbb{X}_{[-\infty, n]}$ is generated by $\star$.
- Assume true for $n = \alpha$.
- Assuming true for n

$$\begin{tikzcd}
\coprod \Sigma^{n} \star \arrow[r] \arrow[d] & \mathbb{X}_{[-\infty, n]} \arrow[d] \\
\emptyset \arrow[r] & \mathbb{X}_{[-\infty, n+1]} \arrow[ul, phantom, "\ulcorner"]
\end{tikzcd}$$

All but ~~bottom~~ right term are gen. by $\star$, so bottom right term is too (it's a colim diagram).

- Apply for $n = \beta \to \mathbb{X}_{[-\infty, \beta]} = \mathbb{X}$. $\square$

Now for step 2, need to identify endomorphisms

$$\widetilde{\mathrm{Flow}}^{fr}(\star, \star) \simeq \mathbb{S} \text{ as } \underline{\text{ring}} \text{ spectra}$$

<u>Fact</u> $\mathbb{S}$ is a ring in a unique way, so it suffices to show that $\widetilde{\mathrm{Flow}}^{fr}(\star, \star) \simeq \mathbb{S}$ as spectra

We identify 0^{th} spaces: $\mathrm{Flow}^{fr}(\star, \star) \simeq \Omega^{\infty} \mathbb{S}$

+ other spaces are similar.

Recall (Pontjargin–Thom spectrum):

$$\Omega_k^{fr} := \{\text{closed framed } k\text{-mflds}\} / \text{bordism}$$

$$\cong$$

$$\pi_k \mathbb{S} := \lim_{i \to \infty} \pi_{k+i} S^i$$

$\hookrightarrow$ Proof: $F : \pi_k \mathbb{S} \to \Omega_k^{fr}$

$\alpha : S^{k+i} \to S^i$ $\quad [\alpha] \mapsto [M]$, where $M := \alpha^{-1}(p)$ closed k-mfld

get framing from framings on S^k, S^{k+i} etc

$p \in S^i$ reg. value.

Intermediate space.

Recall: homotopy coherent nerve: topological categories $\longrightarrow$ ∞-categories

$\ast$ <u>def</u>: $\mathcal{C}$:= topological category with:
- $\mathrm{Ob}(\mathcal{C})$ = vect sp V
- Maps $V \to W$ is $\{\mathrm{Maps}\,(DV, \partial) \to (DW, \partial)\}$

unit disk; $\partial DV = SV$, $\partial DW = SW$ $\leftarrow$ unit sphere.

$\ast$ <u>def</u>: $\Omega^{\infty}(D, \partial D)^{\infty}$:= htpy coherent nerve of $\mathcal{C}$

Explicitly:
- 0-simplices are vect. spaces
- n-simplices consist of:
  - a sequence $(V_0, \ldots, V_n)$ of objects

$\forall\, 0 \leq i \leq n$ — maps $h_{ik} : (DV_i, \partial) \times [0,1]^{\{i+1, \ldots, k-1\}} \to (DV_k, \partial)$

+ compatibility: for $i < j < k$

$$h_{ik}\big|_{\{j^{th} \text{ co-ord is } 0\}} = h_{jk} \circ h_{ij}$$

- 1-simplex (V_0, V_1)

$h_{01} : (DV_0, \partial V_0) \longrightarrow (DV_1, \partial V_1)$

- 2-simplex: (V_0, V_1, V_2)

h_{01}, h_{12}, h_{02}.

$$\begin{tikzcd}
& (DV_1, \partial V_1) \arrow[dr, "h_{12}"] & \\
(DV_0, \partial V_0) \arrow[ur, "h_{01}"] \arrow[rr, "h_{02}"'] & & (DV_2, \partial V_2)
\end{tikzcd}$$

$h_{02}\big|_{\{1^{st} \text{ co-ord is } 1\}}$

- Face maps compose adjacent h's
- <u>Degeneracy</u> maps insert Id.
- It is an ∞-cat.

$\ast$ <u>def</u>: $\underline{\mathrm{Bordism}}^{fr} \subseteq \Omega^{\infty}(D, \partial D)^{\infty}$ has k-simplices $\{(V_0, \ldots, V_k), \{h_{ij}\}\}$ where all h_{ij} are smooth and 0 a regular value.

<u>Claim</u>: $\mathrm{Bord}^{fr} \subseteq \Omega^{\infty}(D, \partial D^{\infty})$ is a $\simeq$.

$\ast$ <u>def</u>: Map $\underline{\mathrm{Bord}}^{fr}(V, V) \longrightarrow \mathrm{Flow}^{fr}(V, V)$

$$\{(V_0, \ldots, V_n), h_{ik}\} \longmapsto \mathbb{X}$$

$V_0 = V$, $V_n = V$

where $\mathbb{X}_{ik} := h_{ik}^{-1}\{0\}$.

$$\begin{tikzcd}
\Omega^{\infty} \mathbb{S} = \operatorname{hocolim}\limits_{\dim V} \mathrm{Maps}(S^V, S^V) = \operatorname{hocolim}\limits_{V} \Omega^{\infty}(V, V) \\
\operatorname{hocolim}\limits_{V} \mathrm{Bord}^{fr}(V, V) \arrow[u, "\simeq"] \arrow[d, "\simeq"] \\
\mathrm{Flow}^{fr}(\star, \star)
\end{tikzcd}$$

since mflds embed uniquely up to isotopy in V, $\dim V \gg 0$.