Derived orbifolds

Yesterday dOrb^S ← structure

Today: what is a derived orbifold, why do we care?

① Global Kuranishi charts

* def: G cpct Lie gp, T top. mfld w. cont (eff) act$^°$ (faithful), finite stabilizers, E: G-vector bundle over T, s section, G-equivariant.

- metric space M and a "footprint" homeomorphism
$$s^{-1}(0)/G \xrightarrow{\simeq} M$$
We say (G, T, E, s) global Kuranishi chart for M.

* def: (G, T, E, s) gl. K. chart.
- free along $C \subset s^{-1}(0)/G$ if G-action is free on the preimage of C.
- smooth if all the data except for s is smooth
- regular if s is also smooth & $s \pitchfork 0_T$

Ex: M smooth mfld $\longrightarrow (\{e\}, M, M \times \{0\}, 0)$
→ global K. chart for M

$(\mathbb{Z}/2, \underset{\mathbb{R}/\mathbb{Z}}{\overset{}{S^1}}, S^1 \times \mathbb{R}, 0) \rightsquigarrow$ gl. K. chart for $S^1/\mathbb{Z}_2$

$(\text{—— '' ——}, t \mapsto 1 - \exp(8\pi t)) \rightsquigarrow$ chart for $\overset{*}{\mathbb{Z}_2} \quad \overset{*}{\{e\}} \quad \overset{*}{\mathbb{Z}_2}$

[Rmk by Amanda: can always increase the obstruction bundle by changing the section]

Operations: $(G, \underset{\text{thickening}}{T}, \underset{\text{obstruct}^° \text{ bundle}}{E}, \overset{\text{obstruction section}}{s})$

- germ equivalence: $\mathcal{U} \subset T$ open neigh of $s^{-1}(0)$
$(G, T, E, s) \rightsquigarrow (G, \mathcal{U}, E|_{\mathcal{U}}, s|_{\mathcal{U}})$ restrict$^°$
- stabilisation: $W \xrightarrow{p} T$ G vb
$(G, T, E, s) \rightsquigarrow (G, W, p^*E \oplus p^*W, p^*s \oplus \Delta)$
- group enlargement: G' cpt Lie gp, $P \xrightarrow{q} T$ G-eq princ G'-bundle

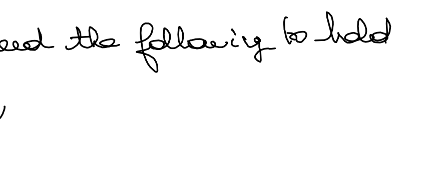
If we have a global Kuranishi chart $\xrightarrow{\text{pass to quotient}}$ obtain a derived orbifold.

* def: $\mathbb{X} = (X, T^-\mathbb{X}, s)$ is a derived orbifold
X: orbifold (no corners); $T^-\mathbb{X}$: vb over X; s: section of $T^-\mathbb{X}$

* def: $T\mathbb{X} = (T^+X, T^-\mathbb{X})$ tangent bundle of $\mathbb{X}$
T^+X: tangent bundle of X

* Rmk: (G, T, E, s) global Kuranishi chart.
$\Rightarrow (T/G, E/G, s)$ derived orbifold.

* ex: $\underset{\mathbb{Z}/2}{*} \quad * \quad \underset{\mathbb{Z}/2}{*} \longleftarrow$



Morphisms of derived orbifolds

A map $f: \mathbb{X} \to \mathbb{X}'$ we need the following to hold
- f ("good") map $X \to X'$ — morphism don't worry too much
- $\hat{f}: T^-\mathbb{X} \to f^*T^-\mathbb{X}'$

$$\begin{tikzcd} T^-\mathbb{X} \arrow[r, "\hat{f}"] & f^*T^-\mathbb{X}' \\ X \arrow[u, "s"] \arrow[ur, "f^*s'"'] \arrow[u, phantom, shift left=6, "\circlearrowright"] \end{tikzcd}$$

Strong equivalence:

$\mathbb{X}, \mathbb{X}'$ strongly equivalent if
- $f: \mathbb{X} \to \mathbb{X}'$
- a vb structure on $X' \to X$ st this map f gets identified w. the inclusion of the zero sect$^°$

Ex: $(\mathbb{Z}/2, \underset{z \mapsto -z}{\mathbb{C}}, \underset{(z,v) \mapsto (-z,v)}{\mathbb{C} \times \mathbb{C}}, s)$ Let's take $s(z) = z^2$ $\Big\}$ Quotient $\mathbb{X}'$

- $(\mathbb{Z}/2, \{pt\}, 0, 0) \}$ $\mathbb{X}$

- Note that $\begin{matrix} \mathbb{C} \\ \downarrow \\ \{pt\} \end{matrix}$ is a vb over pt.

$f: \{pt\} \longrightarrow \{0\} \in \mathbb{C}$

on the bundle side:
$$\begin{tikzcd} 0 \arrow[r] & f^*(\mathbb{C} \times \mathbb{C}) = \text{fibre over } 0. \\ \{pt\} \arrow[u] \arrow[ur, "f^*s"'] \end{tikzcd}$$

Correspondence between derived orb & gl. Kuranishi charts.

(Pardon '19) $(G, T, E, s) \longmapsto (T/G, E/G, s)$ derived orb. by passing to quot.

$\exists\, n$ st $(U(n), \mathcal{T}, E, \hat{s})$ gl Kuranishi charts. $\longleftarrow$ NON-TRIVIAL $(X, T^-\mathbb{X}, s)$ w. X cpt

* Rmk: (G, T, E, s) global Kuranishi chart
$\overset{\text{stabilize}}{\rightsquigarrow} (G, W, p^*E \oplus p^*W, p^*s \oplus \Delta)$

gives a strong equivalence on the resp. derived orbifolds.

Why is this interesting to us? Link to Floer theory?

Want global Kuranishi charts for moduli spaces of J-hol curves.

↳ Need orbifolds if there are non-trivial automorphisms of curves, get isotropy groups

↳ Need to be derived because in general transversality fails, obstructions (encoded in lin. op. having cokernel) can be encoded in the obstruction bundle, and we can still present our moduli spaces that way.