1.5.2024

The main results of the Abouzaid-Blumberg papers

Q: What does this paper change concretely for Floer homotopy theory?

I will only talk about motivation, applications come tomorrow.

Previously: Following a suggestion by Floer (!), Cohen-Jones-Segal proposed the following strategy to extract information from higher-dim. moduli spaces:

- build a framed flow category (using gradient flow lines)
- build out of that a CW complex whose stable homotopy type $\rightsquigarrow$ spectrum is independent of choices (if compactness fails $\rightsquigarrow$ pro-spectrum)

But: We are not interested in the CW complex itself. What if we could extract the homotopy-theoretic information we want directly from the flow category? And can we do that in a geometric vs an algebraic way?

For this, we need a sufficiently robust "theory" of flow categories and relations between them.

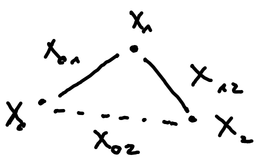
Inspiration: Bordism theory

Thm 1 (AB): (structured) Flow categories are the objects of a stable quasi-category Flow, whose morphisms are (structured) flow bimodules.

Let's unpack this.

1) quasicategory: · we have for $n \geq 0$ a set Flow_n of flow n-simplices with a 0-simplex a flow category

a) flow bimodules form a notion of morphisms between flow categories



b) they can be composed and the composition is unique up to coherent higher homotopies

c) We have mapping spaces (!) $\mathrm{Flow}(X,Y)$ for any X,Y ← since Kan complex

d) Unwinding definitions, we see that

$\pi_k(\mathrm{Flow}(X,Y)) \cong \{k\text{-dim. flow bimodules } X \to Y\}/\text{bordism}$

— $\mathrm{Flow}_*(X,Y) \times \mathrm{Flow}_*(Y,Z) \longrightarrow \mathrm{Flow}_*(X,Z)$

which admits a unit in $\mathrm{Flow}_*(X,X)$.

2) stable: → the objects behave like spectra

⇕ finite admits limits and colimits + behaves like a derived category

· ∃ zero object: flow category with no objects

· ∃ suspension $\Sigma : \mathrm{Flow} \to \mathrm{Flow}$ which is a quasi-equivalence with

$\Omega\,\mathrm{Flow}(X,Y) \cong \mathrm{Flow}(X,\Sigma Y) = \mathrm{Flow}(\Sigma^{-1}X, Y)$

· notion of fibre and cofibre of flow bimodules

Flow has another distinguished object : the unit, denoted $*$, and given by the flow category with a single object.

I imagine, we have to wait for the sequel to see why this is called the unit.

Consequences :

i) Flow is enriched over spectra: We can define for flow categories X, Y a mapping spectrum $\widetilde{\mathrm{Flow}}(X,Y)$ where $\widetilde{\mathrm{Flow}}(X,Y)_n := \mathrm{Flow}(X, \Sigma^n Y)$.

ii) By construction, this is an (Ω-)spectrum, so

$$\pi_k(\widetilde{\mathrm{Flow}}(X,Y)) = \pi_0(\widetilde{\mathrm{Flow}}(\Sigma^k X, Y)) \cong \pi_k(\mathrm{Flow}(X,Y))$$

Ex: The stable homotopy groups of $\widetilde{\mathrm{Flow}}(*, X)$ are exactly the bordism classes of right X-flow modules.

Cor: $\widetilde{\mathrm{Flow}}(*,*)$ is a ring spectrum with homotopy groups

$$\mathrm{Flow}_*(*,*) = \text{bordism groups of (structured) derived orbifolds}$$

Note: These groups vanish if we consider unstructured derived orbifolds since $[-1,1] \times X /_{\mathbb{Z}_2}$ is a null-bordism for any derived orbifold X.

We will from now on focus on framed flow categories whose morphism spaces are manifolds.

Prop: Any non-trivial flow category admits a nontrivial map from $*$, i.e., $\mathrm{Flow}^{fr}_*(*, X) \neq 0$ for any $X = \emptyset$.

abstract nonsense

Prop: Flow^{fr} is equivalent to the category of modules of the spectrum $\widetilde{\mathrm{Flow}}^{fr}(*,*)$.
via the map $X \longmapsto \widetilde{\mathrm{Flow}}^{fr}(*, X)$?

According to the last proposition, we only have to identify the spectrum $\widetilde{\mathrm{Flow}}^{fr}(*,*)$. This uses the Pontryagin-Thom construction.

Thm 2 (AB): The stable quasicategory of framed flow categories is equivalent to the category of spectra.

Cor: The stable homotopy (type) groups of the spectrum associated to the framed flow category X are exactly the bordism groups $\mathrm{Flow}^{fr}_*(*, X)$.

Q: Is this the spectrum associated to X via the CJS construction?