Floer homotopy theory Spring School

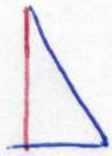
Quasi-category of flow categories (section 5 of the paper)

$\mathrm{Flow}_\bullet$ semi-simplicial set with $\mathrm{Flow}_0$ = flow categories enriched over derived orbifolds with corners.

Theorem (AB): $\mathrm{Flow}_\bullet$ can be endowed with the structure of a quasi-category ie of a simplicial set satisfying the inner horn filling condition.

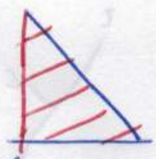
Plan: restrict to flow categories enriched over smooth manifolds with corners

① Filling the missing facet of a horn in $\mathrm{Flow}_\bullet$ (topologically, no smooth structure yet).



(the horn-filling condition can be formulated for semi-simplicial sets).

② Filling a whole inner horn (topologically)



③ Smooth structures on the morphism spaces of a filling

What remains then:

* Filling structured horns
* Filling horns in the setting of derived orbifolds
* Make $\mathrm{Flow}_\bullet$ a simplicial set ie construct degeneracies $\mathrm{Flow}_q \to \mathrm{Flow}_{q+1}$ (eg $d^0$: $\mathrm{Flow}_0 \to \mathrm{Flow}_1$, $d^0(X) = id_X$).

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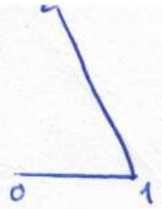
For now: $\mathrm{Flow}_\bullet$ = semi-simplicial set of flow categories enriched over Man = smooth manifolds with corners.

Inner horns in $\mathrm{Flow}_\bullet$

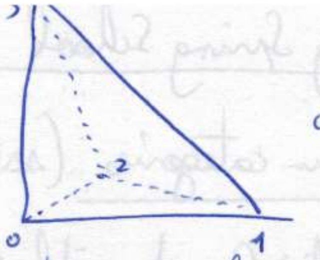
$n \geq 2$, $0 < k < n$ $\Lambda^n_k : \Delta_+^{op} \to \mathrm{Set}$

Geometrically: $\Delta^n$ minus its interior and the face opposite to the vertex $k$.

$n=2, k=1$:




$n=3, k=2$:




no interior and no face at the forefront.

Combinatorially: $(\Lambda^n_k)_i$ = $i$-vertices of the geometric $\Lambda^n_k$-horn.

ex: $(\Lambda^n_k)_{n-1}$ = faces opposite to $i$ for $i \neq k$
has $n$ elements

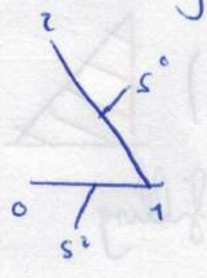
A $\Lambda^n_k$-horn in Flow. is a morphism (i.e. a natural transformation) $\Lambda^n_k \xrightarrow{h} \mathrm{Flow}_\bullet$.

Concretely: $n$ $(n-1)$-simplices in Flow. $s^0, s^1, \dots, s^{k-1}, s^{k+1}, \dots, s^n$ where

$$h_{n-1} : (\Lambda^n_k)_{n-1} \longrightarrow \mathrm{Flow}_{n-1}$$
$$\text{face opposite to } i \longmapsto s^i,$$

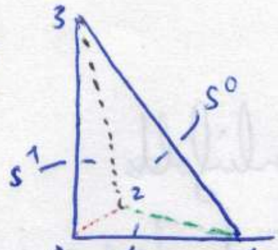
with the condition that they agree on their common faces.

ex: $n=2, k=1$:




$$\partial^1 s^2 = \partial^0 s^0$$

$n=3, k=2$:



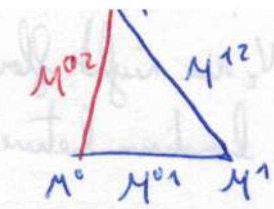

$$\partial^{02} s^3 = \partial^{01} s^1$$
$$\partial^{12} s^1 = \partial^{02} s^0$$
$$\partial^{12} s^3 = \partial^{01} s^0$$

Goal: Show that for such a horn there exists a $n$-simplex $\Delta^n \to \mathrm{Flow}.$ whose $k$-horn is $\Lambda^n_k$ (face opposite to $i$ is $s^i$).

Filling the missing facet

i.e. given an inner horn $\Lambda^n_k \to \mathrm{Flow}.$, construct a $(n-1)$-simplex $s^k : \Delta^{n-1} \to \mathrm{Flow}.$ agreeing with $s^0, s^1, \dots, s^{k-1}, s^{k+1}, \dots, s^n$ on the common faces.

Special cases: $n=2, \ell=1$: (see intro, p. 6)




$M^0, M^1, M^2$ flow categories

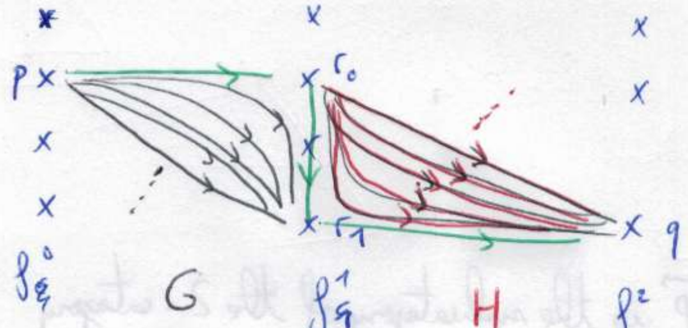
$M^{01}, M^{12}$ flow bimodules

we want to construct $M^{02}$.

We should think of $M^{02}$ as the composition of $M^{01}: M^0 \to M^1$ and $M^{12}: M^1 \to M^2$ in the quasi-category Flow. We take the composite bimodule

2 intuitions: first: formal procedure producing what we want.
second: from Morse theory.

Think of $M^i$ as the flow category of a Morse function $f^i: M \to \mathbb{R}$ and of $M^{01}$ (resp. $M^{12}$) as the flow bimodule arising from a continuation map $G$ between $f^0$ and $f^1$ (resp. $H$ between $f^1$ and $f^2$).




$M^{02}(p,q) = \coprod_{r \in M^1} M^{01}(p,r) \times M^{12}(r,q) /\sim$ where $\sim$ identifies the common boundary strata, ie the images of the maps

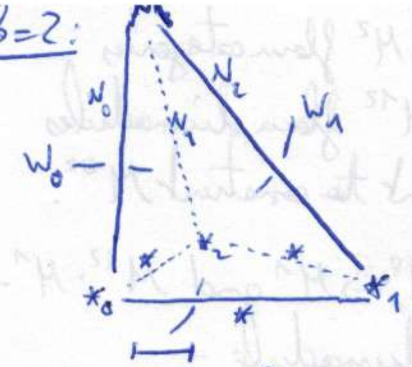
$$M^{01}(p,r_0) \times M^1(r_0,r_1) \times M^{12}(r_1,q)$$

$$\swarrow \qquad\qquad\qquad \searrow$$

$$M^{01}(p,r_1) \times M^{12}(r_1,q) \qquad\qquad M^{01}(p,r_0) \times M^{12}(r_0,q)$$

$\leadsto$ $M^{02}(p,q)$ is the moduli space of concatenation of broken flow lines of $G$ and $H$ connecting $p$ to $q$. At the level of Morse complexes we have $\Phi_{02} = \Phi_{21} \circ \Phi_{01}$.

$n=3, \ell=2$:




$N_0, N_1, N_2$ are right flow modules over $M$
$W_0$ is a bordism between $N_0$ and $N_1$
$W_1$ ——————— $N_1$ and $N_2$.

The value $W_2$ on the missing facet should be a bordism between $N_0$ and $N_2$ obtained by gluing $W_0$ and $W_1$ along $N_1$. It will be:

$$\forall p \in M, \ W_2(p) = \operatorname{colim} \begin{pmatrix} N_1(p) \xhookrightarrow{i_0} N_1(p) \times [0,1] \xhookleftarrow{i_1} N_1(p) \\ \downarrow \qquad\qquad\qquad\qquad \downarrow \\ W_0(p) \qquad\qquad\qquad\qquad W_1(p) \end{pmatrix}.$$

The filling property in that case yields the transitivity of the bordism relation on right flow modules.

General definition:

$\vec{P} = (P_0, \dots, P_n)$ , $0 < \ell < n$

Definition: The 2-category $\Lambda^n_\ell \vec{P}$ is the subcategory of the 2-category $\vec{P}$ with same objects and $\Lambda^n_\ell \vec{P}(p,q)$ = labelled trees in $\vec{P}(p,q)$ which do not contain a vertex labelled either by $\{1, \dots, n-1\}$ or by the complement of $\ell$ in this set.

A $\Lambda^n_\ell$ horn in Flow is then a category $Y$ enriched over manifolds with corners and stratified by $\Lambda^n_\ell \vec{P}$ for some $\vec{P}$.

From such a $Y$, fix $p, q \in \Lambda^n_\ell \vec{P}$ and define a contravariant functor

$$Y_{(-)}(p,q) : \Lambda^n_\ell \vec{P}(p,q) \longrightarrow \mathrm{Man}$$

$$\alpha = \underset{p \ T_0 \ p_1}{\bullet\!\!-\!\!\bullet\!\!-\!\!\bullet} \cdots \underset{p_i \ T_i \ p_{i+1}}{\bullet\!\!-\!\!\bullet\!\!-\!\!\bullet} \cdots \underset{T_r \ q}{\bullet\!\!-\!\!\bullet} \longmapsto \prod_{i=0}^{r} Y\big(\underset{p_i \ T_i \ p_{i+1}}{\bullet\!\!-\!\!\bullet\!\!-\!\!\bullet}\big)(p_i, p_{i+1}).$$

We define the missing face of the horn as:

$$s^\ell(p,q) = \operatorname*{colim}_{\alpha \in \Lambda^n_\ell \vec{P}(p,q)} Y_\alpha(p,q).$$

# Filling a whole inner horn

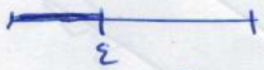
We [redacted] define $\Delta_+$ the subcategory of $\Delta$ [redacted] obtained by keeping only injective maps.

We have a codimension functor

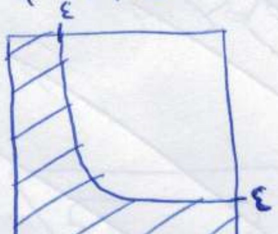
$$\mathrm{codim}: \Lambda^n_k \vec{P}(p,q) \longrightarrow \Delta_+$$

<u>L-blocks</u> fix $0<\varepsilon<1$ and define $L_d \subset [0;1]^{d+1}$ by the set of elements satisfying $0 \le \prod_{i=0}^{d} x_i \le \varepsilon$.

<u>d=0</u>: $L_0 \subset [0;1]$




<u>d=1</u>: $L_1 \subset [0;1]^2$




if we have an injective map $\{0;\dots;k\} \xrightarrow{\varphi} \{0;\dots;d\}$ we can set the elements $x_i$ for $i \notin \mathrm{im}\,\varphi$ equal to 1, and we get an inclusion $L_k \subset L_d$.

This makes $d \mapsto L_d$ a functor $\Delta_+ \longrightarrow \mathrm{Man}$.

<u>Definition</u>: The value of the horn filling is given by

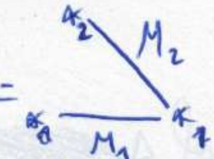
$$Y(p,q) = \coprod_{\substack{\alpha \to \beta \\ \in \Lambda^n_k \vec{P}(p,q)}} L_{\mathrm{codim}(\alpha)} \times Y_\beta(p,q) /\sim$$

where $\sim$ identifies the images of the maps

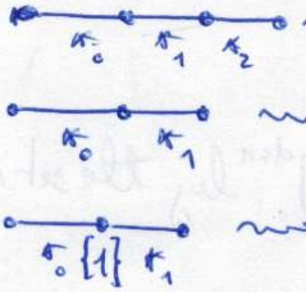
$$L_\beta \times Y_\beta(p,q) \longleftarrow L_\alpha \times Y_\beta(p,q) \longrightarrow L_\alpha \times Y_\alpha(p,q).$$

Examples:

$q=1, n=2$:

$Y=$ 

we get $Y(x_0,x_2) = \cancel{(M_1\times M_2)\times[0}\ \underbrace{[0;1]}_{=L_0}\times M_1\times M_2$ w/ stratification given by:

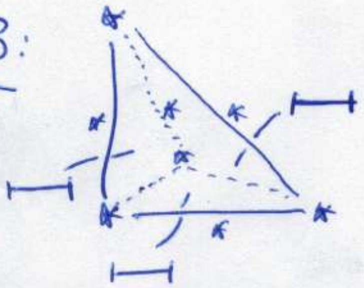


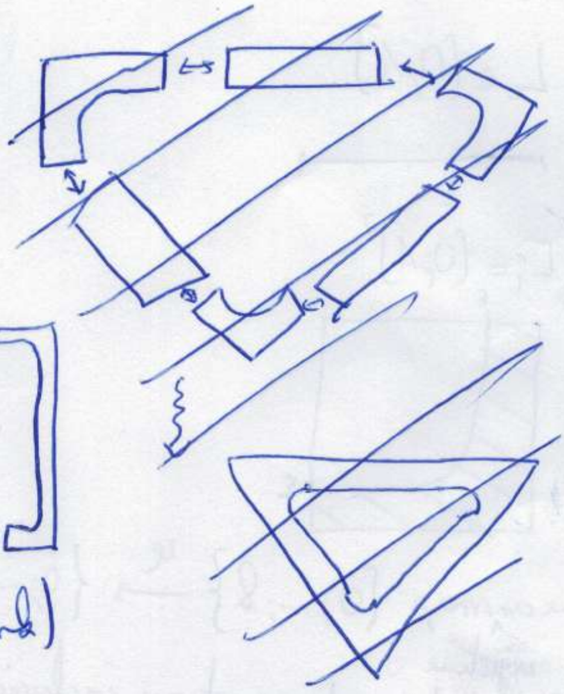
$\rightsquigarrow 0\times M_1\times M_2$

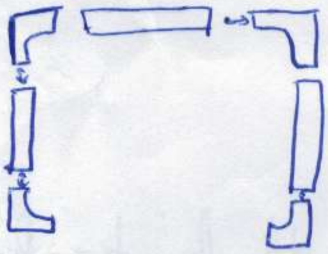
$\rightsquigarrow 1\times M_1\times M_2$

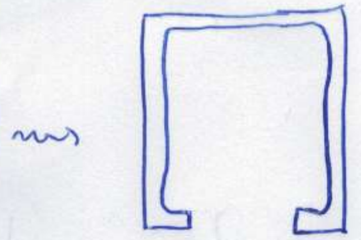
$\rightsquigarrow [0;1]\times M_1\times M_2$

$q=2, n=3$:









(I think)

Smooth structure:

Recall if we have two manifolds w/ boundary $M, N$ and a diffeo $\partial M \xrightarrow{\varphi} \partial N$ then a choice of collars i.e. diffeos

$c_M : \partial M \times [0;1[ \longrightarrow$ neighborhood of $\partial M$ in $M$, $(c_M)_{|\partial M} = id$

$c_N : \partial N \times ]-1;0] \longrightarrow$ ——— $\partial N$ in $N$, $(c_N)_{|\partial N} = id$

define a homeo between a neighborhood of $\partial M$ in $M\cup_\varphi \partial N$ and $\partial M\times ]-1;1[$, which endows $M\cup_\varphi N$ w/ a smooth structure such that $N, M \hookrightarrow M\cup_\varphi N$ are smooth.

Rk: depends on choice of collars! OK for us since we don't require unicity of fillings.

Here: choose collars on $L_\alpha\times Y_\beta(p,q)$ for all $\alpha\longrightarrow\beta$

This identifies a neighborhood of $L_\alpha\times Y_\beta(p,q)$ in $Y(p,q)$ w/ $L_\alpha\times Y_\beta(p,q)\times ]-1;1[^\delta$ where $\delta = \text{codim}\,\beta - \text{codim}\,\alpha$.