Flow^{fr} IS EQUIVALENT TO SPECTRA

The aim of this talk is to prove the following theorem.

Theorem 0.1 (Abouzaid-Blumberg)**.** *The ∞-category Flow^{fr} of framed flow categories is equivalent to the category of spectra.*

Recall that we showed that Flow^{fr} is a stable ∞-category with a distinguished object, called the *unit* and denoted $*$, given by the flow category with a single object and no morphisms. Throughout this talk we assume

i) all flow simplices are enriched over frame (derived) manifolds,
ii) we only have zero energy, i.e., $\Gamma = \{0\}$.

As an application, we note the following result.

Theorem 0.2 (Abouzaid-Blumberg)**.** *Let (X, ω) be a symplectically aspherical symplectic manifold so that $TX \cong E \otimes_{\mathbb{R}} \mathbb{C}$ for some real vector bundle $E \to X$. Then to any Hamiltonian H on X there is an associated framed flow category $HF(H; \mathrm{Flow}^{fr})$.*

By Theorem 0.1, we thus obtain a spectrum. Such a spectrum was previously constructed by Cohen-Jones-Segal and Large.

The proof has two steps (corresponding to the two subsections of §8 of the paper:

1. Flow^{fr} is generated by the unit.
2. The mapping spectrum $\widetilde{\mathrm{Flow}}^{fr}(*, *)$ is equivalent to the sphere spectrum $\mathbb{S}$ as a ring spectrum.

We first sketch the proof of the first assertion. Here, "generated by"' means that a flow category $\mathbb{X}$ is equivalent the a homotopy colimit of some diagram constant at the unit. For the sake of simplicity we will only discuss the case of flow category with finitely many objects.

Suppose thus, $\mathbb{X}$ is a such a flow category and additionally that there exists a degree function $|\cdot|\colon \mathrm{obj}(\mathbb{X}) \to \mathbb{Z}$. Given $n \in \mathbb{Z}$, we define $X_{(\infty,n]}$ be the flow category given by restricting $\mathbb{X}$ to $\{x \in \mathrm{obj}(\mathbb{X}) \mid |x| \leqslant n\}$.

Proposition 0.3. *There are morphisms $\mathbb{X}_{(-\infty,n]} \to \mathbb{X}_{(-\infty,n+1]} \to \bigsqcup_{|x|=n+1} \Sigma^{n+1} *$ forming a cofibre sequence.*

Proof. The first map is given by the extension of the diagonal bimodule $s\mathbb{X}_{(-\infty,n]}$, while the second is given by the restriction of the diagonal bimodule $s\mathbb{X}_{(-\infty,n+1]}$. (For this we use that for any bimodule $\mathbb{B}\colon \mathbb{X} \to \mathbb{Y}$ between graded flow categories we have that $\mathbb{B}(p, q) = \varnothing$ if $|p| \leqslant |q| + 1$.) $\square$

Let $a < b$ be such that $a < |x| \leqslant b$ for any $x \in \mathrm{obj}(\mathbb{X})$. In other words, $X_{(-\infty,a]} = \varnothing$ and $X_{(-\infty,b]} = \mathbb{X}$. We will inductively show that $X_{(-\infty,n]}$ is generated by the unit. This is clear for $n \leqslant a$, using the empty diagram. Suppose thus it holds for some n. Using the long exact sequence of the cofibre sequence in Proposition 0.3, we obatin a (co)fibre square

$$\begin{array}{ccc} \bigsqcup_{|x|=n+1} \Sigma^{n} * & \longrightarrow & \mathbb{X}_{(-\infty,n]} \\ \downarrow & & \downarrow \\ \varnothing & \longrightarrow & \mathbb{X}_{(-\infty,n+1]} \end{array}$$

Since all flow categories in the square except $\mathbb{X}_{(-\infty,n+1]}$ are generated by the unit and the square is a pushout square, $\mathbb{X}_{(-\infty,n+1]}$ is generated by the unit as well.

We now turn to the second step in the proof. We first note

Fact: $\mathbb{S}$ admits a unique ring spectrum structure.

Therefore, we only have to identify $\widetilde{\text{Flow}}^{fr}(*,*)$ with $\mathbb{S}$ as spectra. The key ingredient of this proof is a slightly more involved version of the Pontryagin-Thom construction, which we briefly recall.

Theorem 0.4. $\Omega_*^{fr} = \pi_*(\mathbb{S})$.

Proof. Define $F\colon \pi_k(\mathbb{S}) \to \Omega_k^{fr}$ by $F(\alpha) = [f^{-1}(\{x\}), \Phi_{f,x}]$, where $f\colon S^{n+k} \to S_n$ is any smooth function representing α, x is a regular value of f and $\Phi_{f,x}$ is any framing of the normal bundle of $f^{-1}(\{x\})$ in S^{n+k} given by the identification with $f^{-1}(\{x\}) \times T_xS^n$ (which stably can be identified uniquely with a framing of $Tf^{-1}(\{x\})$). It is straightforward to check that F is well-defined. Conversely, we define $G\colon \Omega_k^{fr} \to \pi_k(\mathbb{S})$ by choosing for a closed k-manifold M with an isomorphism $\Phi\colon TM \oplus \mathbb{R}^\ell \to M \times \mathbb{R}^{k+\ell}$ an embedding $\iota\colon M \hookrightarrow \mathbb{R}^n$ for $n \gg 1$. If $\mathcal{N}$ denotes the normal bundle of ι, we obtain an isomorphism

$$\mathcal{N} \oplus \mathbb{R}^{k+\ell} \xrightarrow{\text{id}\oplus\Phi^{-1}} \mathcal{N} \oplus TM \oplus \mathbb{R}^\ell \cong M \times \mathbb{R}^{n+\ell}.$$

On the other hand, we have a tubular neighbourhood $\mathcal{N} \xrightarrow{\psi} U \subset \mathbb{R}^n$. Collapsing the complemenet of $U \times \mathbb{R}^{k+\ell}$ to the point at infinity, we obtain a continuous map g given by the composite

$$S^{n+k+\ell} \to U \times \mathbb{R}^{k+\ell}/\partial U \times \mathbb{R}^{k+\ell} \to \text{Th}(\mathcal{N} \times \mathbb{R}^{k+\ell}) \to \text{Th}(M \times \mathbb{R}^{n+\ell}) \to S^{n+\ell}.$$

We set $G([M, \Phi]) = [g]$. Again, it is a long but straightforward computation to verify that G is well-defined and an inverse of F. $\square$

We now lift to to prove the second step. To this end, let $\mathcal{C}$ be the category enriched over topological spaces with objects being given by inner product spaces and whith morphisms

$$\mathcal{C}(V, W) = C((DV, SV), (DW, SW)),$$

where DV denotes the unit ball and SV the unit sphere of V respectively. Let $\Omega^\infty(D, \partial D)$ be the homtopy coherent nerve of $\mathcal{C}$. Explicitly, we have

$$\Omega^\infty(D, \partial D)_0 = \text{obj}(\mathcal{C})$$

and for $n > 0$, we let

$$\Omega^\infty(D, \partial D)_n \subset \Big\{((V_0, \dots, V_n), \{h_{ij}\colon (DV_i, SV_i) \times [0,1]^{\{i+1,\dots,j-1\}} \to (DV_j, SV_j)\})\Big\}$$

given by tuples so that for all $0 \leqslant i < j < k \leqslant n$ we have

$$h_{ik}|_{(DV_i,SV_i)\times[0,1]^{\{i+1,\dots,j-1\}}\times\{0\}\times[0,1]^{j+1,\dots,k-1}} \simeq h_{jk} \circ h_{ij}.$$

Then 1-simplices correspond to maps $(DV_0, SV_0) \to (DV_1, SV_1)$ and two simplices correspond to homotopies from $h_{12} \circ h_{01}$ to h_{02}. The face maps compose adjacent h's, while the degenreacy maps insert identities.

Fact: We have

$$\Omega^\infty\mathbb{S} \cong \text{hocolim}_V Map(S^V, S^V) \cong \text{hocolim}_V \Omega^\infty(V, V),$$

where the last space is the mapping space of Ω^∞ from V to V.

Define $\mathrm{Bord}^{fr} \subset \Omega^\infty(D, \partial D)$ to be the subsimplicial set with n-simplices of the form $((V_0, \ldots, V_n), \{h_{ij}\})$ where each h_{ij} is smooth has 0 as regular value.

Lemma 0.5. *The inclusion* $\mathrm{Bord}^{fr} \hookrightarrow \Omega^\infty(D, \partial D)$ *is an equivalence.*

Lemma 0.6. *The map* $\mathrm{Bord}^{fr}(V, V) \to \mathrm{Flow}^{fr}(V, V)$ *given by*

$$((V_0, \ldots, V_n), \{h_{ij}\}) \mapsto \mathbb{X} = (h_{ij}^{-1}(\{0\}))_{i<j}.$$

is an equivalence. Here we identify the vector space V *with* $\Sigma^{\dim(V)}*$.

Combining these two lemmas with the fact above, we conclude that

$$\Omega^\infty \mathbb{S} \cong \mathrm{hocolim}_V \mathrm{Flow}^{fr}(V, V) = \mathrm{Flow}^{fr}(*, *).$$