Introduction to flow categories

Motivat°: geometric data Floer homology $\rightsquigarrow$ (stable ~~htp~~ htpy type

CJS (1995): use a "framed flow category" ~~to build a~~ ~~spectrum~~

What ~~are~~ is a (framed) flow category?

Roughly, objects are pts in space & morphisms btw them are ~~the~~ "trajectories" btw pts.

+ framing gives data on how the moduli spaces "glue together" rough

$\longrightarrow$ Can ~~recover~~ get chain cplx + CW cpx from flow category

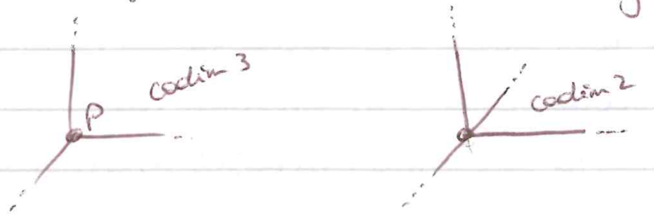
If you know what this means

Stable htpy type is suspension spectrum of CW cplx.

① Definitions

mflds w. corners, locally $(\mathbb{R}_+)^k \times (\mathbb{R})^{k'}$

$p = 0$



etc $\qquad 0 \in \mathbb{R}^3$ codim 0

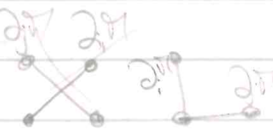
$\partial M = \bigcup$ faces of M, face of M = closure of conn comp of ~~codi~~ set of codim 1 pts.

Mfld with corners + well-behaved boundary.

* def: A smooth m-dim $\langle m \rangle$-mfld is a mfld w. corners M + ~~n-tu~~ $(\partial_1 M, \dots, \partial_m M)$ st:

(i) $\partial_1 M \cup \dots \cup \partial_m M = \partial M$

(ii) $\partial_i M \cap \partial_j M$ ~~is a~~ face of $\partial_i M$ & $\partial_j M$



Spaces in which you want to embed your $\langle n \rangle$-mflds

* Notation: ~~E^d~~ $d = (d_0, \dots, d_{m-1})$ ~~...~~

$\mathbb{E}^d = \mathbb{R}^{d_0} \times \mathbb{R}_+ \times \mathbb{R}^{d_1} \times \mathbb{R}_+ \dots \times \mathbb{R}_+ \times \mathbb{R}^{d_{m-1}}$

$\partial_i \mathbb{E}^d$ is obtained by "i^{th} copy of $\mathbb{R}_+ \overset{\text{rep by}}{\rightsquigarrow} \{0\}$"

$\mathbb{E}^d[a:b] = \mathbb{R}^{d_a} \times \mathbb{R}_+ \times \dots \times \mathbb{R}_+ \times \mathbb{R}^{d_{b-1}}$.

Need cdt° on the bdry

* **def**: A neat ~~embedding~~ immersion of a smooth ~~mfld~~ m-dim $\langle n\rangle$-mfld M is a smooth ~~emb~~ imm.

$$\iota : M \looparrowright \mathbb{E}^d \quad \text{st} \quad \text{(i) } \iota^{-1}\partial_i \mathbb{E}^d = \partial_i M$$

$$\text{(ii) } \partial_I M \pitchfork \partial_J \mathbb{E}^d \text{ transverse}$$

$$I \subset J \subset \{1, \dots, n\} \qquad \partial_J'' M$$

* **Rmk**: Analog of Whitney emb thm: any smooth m-dim $\langle n\rangle$-mfld admits neat emb in some $\mathbb{E}^d$.

\+ neat immersion can always be upgraded to neat emb $(d' \geq d)$

* **def**: A flow category $\mathcal{C}$ is a category with $\mathrm{Ob}(\mathcal{C})$ finite & a grading $|\cdot| : \mathrm{Ob}(\mathcal{C}) \to \mathbb{Z}$ st

- $\hom(x,x) = ~~\emptyset~~ \{id\}$
  $\hom(x,y)$ is a smooth $(|x|-|y|-1)$-dim $\langle |x|-|y|-1\rangle$-mfld $\mathcal{M}(x,y)$
- compos° $\circ : \mathcal{M}(x,z) \times \mathcal{M}(z,y) \to \mathcal{M}(x,y)$ is a smooth embedding in $\partial_i \mathcal{M}(x,y)$ if $i = |x| - |z|$.

  Furthermore: $\circ^{-1}\left(\partial_i \mathcal{M}(x,y) \cap \partial_j \mathcal{M}(x,y)\right) = \begin{cases} \partial_j \mathcal{M}(z,y) \times \mathcal{M}(x,z) & i<j \\ \mathcal{M}(z,y) \times \partial_{i-j} \mathcal{M}(x,z) & \end{cases}$
- $\circ$ induces a diffeo $\partial_i \mathcal{M}(x,y) \cong \coprod_{z,\,|z|=|x|+i} \mathcal{M}(x,z) \times \mathcal{M}(z,y)$ $j>i$

$\mathcal{M}(x,y)$: moduli space from x to y, $\mathcal{M}(x,x) = \emptyset$.

Assume from now on $\min_x \{|x|\} = 0$, $\max_x \{|x|\} = k$

Want to embed all the $\mathcal{M}(x,y)$ into some $\mathbb{E}^d$, $d = (d_0, \dots, d_{k-1})$

$D := d_0 + \dots + d_{k-1}$

put them all in the same space in a way that respects boundaries.

* **def**. A neat imm. emb. ι of a flow cat. $\mathcal{C}$ relative to d is a collect° of neat imm. emb. $\iota_{x,y} : \mathcal{M}(x,y) \looparrowright \mathbb{E}^d[|y| : |x|]$ st $\forall x, y, z \in \mathrm{Ob}(\mathcal{C})$, $(p,q) \in \mathcal{M}(x,z) \times \mathcal{M}(z,y)$:

$$\iota_{xy}(q \circ p) = (\iota_{xz}(p), 0, \iota_{zy}(q))$$

* **def**: A **framing** φ of $(\mathscr{C}, \iota)$ is a collection of immersions
$\varphi_{xy} : \mathcal{M}(x,y) \times [-\varepsilon, \varepsilon]^A \hookrightarrow \mathbb{E}^d[|y| : |x|]$ extending $\iota_{x,y}$, $A = d_{|y|} + \dots + d_{|x|-1}$
st $\forall x, y, z \in \mathrm{Ob}(\mathscr{C})$, $(p,q) \in \mathcal{M}(x,z) \times \mathcal{M}(z,y)$,

$B = d_{|z|} + \dots + d_{|x|-1}$

$$\varphi_{xy}(q \circ p, t_1, \dots, t_A) = (\varphi_{xz}(p, t_1, \dots, t_B), 0, \varphi_{zy}(q, t_{B+1}, \dots, t_A))$$

* **def**: A **framed flow category** is a triple $(\mathscr{C}, \iota, \varphi)$

(2) Geometric data

framed flow cat. $(\mathscr{C}, \iota, \varphi) \longrightarrow$ chain cplx $C_*(\mathscr{C}, \iota, \varphi)$ (a)
$\longrightarrow$ CW cplx $|\mathscr{C}|$ (b)

Cellular chain cplx of $|\mathscr{C}|$ is actually $C_*(\mathscr{C})$.

Chain cplx

(a) $C_i(\mathscr{C}) := \bigoplus_{|x|=i} \mathbb{Z} \cdot x$, ∂x : coeff in y given by (signed) count of pts in $\mathcal{M}(x,y)$
0-dim bc $|y| = |x| - 1$.

$\rightsquigarrow$ "$(\mathscr{C}, \iota, \varphi)$ refines the chain cplx $C_*(\mathscr{C})$"

CW cplx

(b) Construct° of CW complex : Lipschitz-Sarkar ($\neq$ CJS)
(i) Start with one 0-cell
(ii) $\forall x \in \mathrm{Ob}(\mathscr{C})$, associate cell of dim $|x| + D$, $\mathscr{C}(x)$
Choose $\varepsilon > 0$ small & $R > 0$ big st $\forall |y| < |x|$ φ_{xy} embeds $\mathcal{M}(x,y) \times [-\varepsilon, \varepsilon]^{d_{|y|} + \dots + d_{|x|-1}}$
in $[-R,R]^{d_{|y|}} \times [0,R] \times \dots \times [0,R] \times [-R,R]^{|x|-1} \subseteq \mathbb{E}^d[|y| : |x|]$

⚠ $\neq$ objects require $\neq$ copies!

Then $\mathscr{C}(x) := [-R,R]^{d_0} \times [0,R] \times \dots \times [-R,R]^{d_{|x|-1}} \times \{0\} \times [-\varepsilon, \varepsilon]^{d_m} \times \dots \times \{0\} \times [-\varepsilon, \varepsilon]^{d_?}$
$\hookrightarrow$ "Contains" all the moduli spaces $\mathcal{M}(x,y)$ for $|y| < |x|$ via φ_{xy}

(iii) Assume we have built $|\mathscr{C}|^{D+m-1}$, $|x| = m$.
Define attaching maps $\pi_{xy} : \partial \mathscr{C}(x) \to \mathscr{C}(y) / \partial \mathscr{C}(y) \subseteq |\mathscr{C}|^{D+m-1}$

$\forall y \in \mathrm{Ob}(\mathscr{C})$, $|y| < |x|$, we have $\mathscr{C}_y(x) := [0,R] \times [-R,R]^{d_0} \times \dots \times [-R,R]^{d_{|y|-1}} \times \{0\}$
$\times \mathrm{im}(\varphi_{xy} : \mathcal{M}(x,y) \times [-\varepsilon, \varepsilon]^{d_{|y|} + \dots + d_{|x|-1}}) \times \{0\} \times [-\varepsilon, \varepsilon]^{d_{|x|}} \times \{0\} \times \dots$
$\times [-\varepsilon, \varepsilon]^{d_{?-1}}$

$\rightsquigarrow$ Canonically isom to $\mathcal{C}(y) \times \mathcal{M}(x,y)$.
On $\mathcal{C}_y(x) \subseteq \partial \mathcal{C}(x)$, def π as proj to $\mathcal{C}(y)$.
On $\partial\mathcal{C}(x) \setminus \bigcup_y \mathcal{C}(y)$, map everything to base pt.
$\rightarrow$ Construct indep of ε, R, & by homotopies of ι, φ.

Thm: The cellular chain cplx of $|\mathcal{C}|$ is $C_*(\mathcal{C})$. (+ shift by D)

$\hookrightarrow$ Proof (roughly): differential counts # preimages of maps $\pi_{xy}: \partial\mathcal{C}(x) \to \mathcal{C}(y)/\partial\mathcal{C}(y)$, $|y| = |x| - 1$.
Take $p \in \operatorname{int}(\mathcal{C}(y))$.
$\pi_{xy}^{-1}(p) = \#$ pts in $\mathcal{M}(x,y)$ (up to $\pm$ signs!)